SELECTED METHODS OF POTENTIAL THEORY (PROBABILITY MINI-COURSE)

KRZYSZTOF BOGDAN (LECTURES) AND KAROL SZCZYPKOWSKI (PROBLEM SESSIONS)

Abstract. In three lectures and three problem sessions we present a handful of recent results and techniques, which proved to be useful in Harmonic Analysis, Probability and PDEs. This is an extended version of a mini-course given 2-4 Oct at Potential Theory Workshop: Intersections in Harmonic Analysis, Partial Differential Equations and Probability, September 28-October 6, 2023, CIMAT, Guanajuato. The prerequisites for the course are just integration and general exposure to Analysis (limits, series and estimates), but some familiarity of the reader with semigroups of (sub-Markovian) operators, their generators, and quadratic form or with Markov processes, their transition kernels, exit times, Green kernels, and harmonic functions would be very helpful.

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1. Hardy inequality and ground-state representation

This Lecture 1 is based on [11], but we also like to mention [17], [18], [21].

1.1. **Goals and motivation.**

We construct explicit supermedian functions for symmetric sub-Markov semigroups to obtain *Hardy inequality* or *ground-state representation* (Hardy identity) for their quadratic forms.

Hardy inequalities are important in harmonic analysis, potential theory, functional analysis, partial differential equations and probability. In PDEs they are used to obtain a priori estimates, existence and regularity results [44] and asymptotic behaviour of solutions [50]. In functional and harmonic analysis they yield embedding theorems and interpolation theorems, e.g., Gagliardo–Nirenberg interpolation inequalities [41], etc.

The connections of Hardy inequalities to potential theory are well known; see, e.g., [1], [33], [10], [29]. A general rule stemming from the work of Fitzsimmons [33] is this: If $\mathcal{L}$ is the generator of a symmetric Dirichlet form

$\mathcal{E}$, $h \geq 0$ and $\mathcal{L}h \leq 0$, then $\mathcal{E}(u,u) \geq \int u^2(-\mathcal{L}h/h)$. Below we make a similar connection in the setting of *symmetric transition densities* p. When p is integrated against increasing weight in time and any weight in space, we obtain a *supermedian* function h. We also get a weight, q, an analogue of the Fitzsimmons' ratio $-\mathcal{L}h/h$, which yields the Hardy identity or inequality. Our approach is straightforward and general, the resulting Hardy identity of inequality is automatically valid on the whole of the reference L^2 space, and is optimal in some cases.

We simultaneously prove non-explosion results for Schrödinger perturbations $\tilde{p}$ of p by q. Namely, we verify that h is supermedian and integrable for $\tilde{p}$, if finite. For instance, we recover the famous critical non-explosion result of Baras and Goldstein for $\Delta + (d/2-1)^2|x|^{-2}$; see [3], [46].

The plan of Lecture 1 is as follows. In Theorem 1.6, we prove a non-explosion for Schrödinger perturbations. In Theorem 1.9, we prove a Hardy inequality, in fact, under mild additional assumptions, Hardy identity (ground-state representation), with an explicit remainder term. Then we present applications: the classical Hardy inequality and ground-state representation for the Laplacian and fractional Laplacian. In particular, we recover the optimal constants and the corresponding remainder terms, as given by Filippas and Tertikas [32], Frank, Lieb, and Seiringer [34], and Frank and Seiringer [35].

Current applications of our methods involve detailed analysis of "critical" Schrödinger perturbations and some analogues in the L^p setting; see [17], [21], and [18], respectively.

1.2. **Notation.** Throughout we use ":=" or *cursive* to indicate definition or something noteworthy, e.g., $a \wedge b := \min\{a,b\}$ and $a \vee b := \max\{a,b\}$ for (real numbers) $a, b \in \mathbb{R}$. For real number or real-valued function f, we let $f_+ := f \vee 0$. *Positive* means ≥ 0, analogously for *increasing* (and *decreasing*). For two positive functions f and g we write $f \approx g$ if there is a *strictly positive* number c, called *constant*, such that $c^{-1}\, g \leq f \leq c\, g$. Such comparison is called *sharp estimate*. We write $c = c(a, b, \ldots, z)$ to claim that c may be so chosen to depend only on $a, b, \ldots, z$. Symbols for constants may denote different numbers in different places. For an open subset D of the d-dimensional Euclidean space $\mathbb{R}^d$, $d \in \{1, 2, \ldots\}$, we denote by $C_c(D)$ the space of continuous functions with compact supports in D, and by $C_c^\infty(D)$ the space of infinitely often differentiable functions in $C_c(D)$. The Lebesgue measure on the half-line $[0,\infty)$ is usually denoted by ds, dt, etc., and on $\mathbb{R}^d$, by dx, dy, etc. As usual, $0 \cdot \infty := 0$. Further notation is introduced as we proceed.

1.3. **The Gaussian kernel.** Let g be the Gaussian kernel

$$g_t(x) := (4\pi t)^{-d/2} e^{-|x|^2/(4t)}, \quad t > 0, \quad x \in \mathbb{R}^d. \tag{1.1}$$

Below, as usual, $f * h(x) := \int_{\mathbb{R}^d} f(x-y)h(y)dy$, $x \in \mathbb{R}^d$, the convolution of functions $f, h : \mathbb{R}^d \to \mathbb{R}$, defined if the integral is convergent.

Exercise 1.1. Prove that the function $p_t(x,y) := g_t(y-x)$, $t > 0$, $x, y \in \mathbb{R}^d$, is symmetric: $p_t(x,y) = p_t(y,x)$, and satisfies the Chapman–Kolmogorov equations:

$$\int_{\mathbb{R}^d} p_s(x,y)p_t(y,z)dy = p_{s+t}(x,z), \qquad x, z \in \mathbb{R}^d,\ s, t > 0.$$

In short, $p_t(x,y)$ is a *transition density* on $\mathbb{R}^d$. Further, $\int_{\mathbb{R}^d} p_t(x,y)dy = 1$ for $x \in \mathbb{R}^d$, $t > 0$, so $p_t(x,y)$ is a *probability transition density.*

Hints. By Tonelli's theorem, we verify that $\int_{\mathbb{R}^d} g_t(x)dx = 1$, $t > 0$, after first considering $d = 2$ and polar coordinates. Then we verify that $g_s * g_t = g_{s+t}$, $s, t > 0$, by *reorganizing squares* in the exponents on the left-hand side. The results for p are reformulations.

1.4. **Supermedian functions.** Let $(X, \mathcal{M}, m)$ be a σ-finite measure space. Let $\mathcal{B}_{(0,\infty)}$ be the Borel σ-field on the half-line $(0,\infty)$. Let $p : (0,\infty) \times X \times X \to [0,\infty]$ be $\mathcal{B}_{(0,\infty)} \times \mathcal{M} \times \mathcal{M}$-measurable and symmetric:

$$p_t(x,y) = p_t(y,x), \quad x, y \in X, \quad t > 0.$$

Let p satisfy the Chapman–Kolmogorov equations:

$$\int_X p_s(x,y)p_t(y,z)m(dy) = p_{s+t}(x,z), \qquad x, z \in X,\ s, t > 0, \tag{1.2}$$

and assume that for all $t > 0$ and $x \in X$, $p_t(x,y)m(dy)$ is a σ-finite measure.

Let $f : \mathbb{R} \to [0,\infty)$ be increasing and $f := 0$ on $(-\infty, 0]$. We have $f' \geq 0$ almost everywhere (a.e.), and

$$f(a) + \int_a^b f'(s)ds \leq f(b), \quad -\infty < a \leq b < \infty. \tag{1.3}$$

Further, let μ be a positive σ-finite measure on $(X, \mathcal{M})$. We put

$$p_s\mu(x) := \int_X p_s(x,y)\,\mu(dy), \tag{1.4}$$

$$h(x) := \int_0^\infty f(s) p_s\mu(x)\,ds. \tag{1.5}$$

We also denote $p_t h(x) := \int_X p_t(x,y)h(y)m(dy)$. By Tonelli and Chapman-Kolmogorov, for $t > 0$ and $x \in X$,

$$\begin{aligned} p_t h(x) &= \int_t^\infty f(s-t) p_s\mu(x)\,ds \\ &\le \int_t^\infty f(s) p_s\mu(x)\,ds \\ &\le h(x). \end{aligned} \tag{1.6}$$

In this sense, h is *supermedian* for the *kernel* p. In fact, it is *excessive* since $p_t h \to h$ as $t \to 0$; see [37] for some nomenclature of potential theory.
We then define $q : X \to [0, \infty]$ as follows: $q(x) := 0$ if $h(x) = 0$ or ∞, else

$$q(x) := \frac{1}{h(x)} \int_0^\infty f'(s) p_s\mu(x)\,ds.$$

Hence for all $x \in X$,

$$q(x)h(x) \le \int_0^\infty f'(s) p_s\mu(x)\,ds. \tag{1.7}$$

Exercise 1.2. Calculate h and q for the Gaussian semigroup, μ the Dirac measure, and $f(t) := t^\beta$. For which β we get (the largest) $q(x) = \frac{(d-2)^2}{4}|x|^{-2}$?

Hints. We consider $-\infty < \delta < d/2 - 1$ and calculate the following integral for the Gaussian kernel by substituting $s = |x|^2/(4t)$,

$$\begin{aligned} h(x) &:= \int_0^\infty g_t(x) t^\delta\,dt = \int_0^\infty (4\pi t)^{-d/2} e^{-|x|^2/(4t)} t^\delta\,dt \\ &= (4\pi)^{-d/2} \left(\frac{|x|^2}{4}\right)^{\delta - d/2 + 1} \int_0^\infty s^{d/2-\delta-2} e^{-s}\,ds \\ &= 4^{-\delta-1}\pi^{-d/2}|x|^{2\delta-d+2}\Gamma(d/2-\delta-1). \end{aligned} \tag{1.8}$$

Here, as usual, $\Gamma(p) := \int_0^\infty x^{p-1}e^{-x}dx$, $p \in (0,\infty)$, and $\Gamma(p+1) = p\Gamma(p)$. Optimal $\beta = (d-2)/2$ is obtained by solving a quadratic equation.

1.5. **Schrödinger perturbation.** Two *things* (definitions, theorems, cars, etc.) may be considered *similar* if one seems a special case of another or both seem special cases of a third thing. Here is an instance: exp and $\tilde{p}$ below.

Exercise 1.3. Of course, $\exp(x) := \sum_{n=0}^{\infty} x^n/n!$ for $x \in \mathbb{R}$. Prove directly that $\exp(x+y) = \exp(x)\exp(y)$, $x, y \in \mathbb{R}$.

Hints. This *multiplicativity* follows from simple properties of the binomial symbol $\binom{n}{k}$.

Definition 1.4. [15] We define the Schrödinger perturbation of our p by q:

$$\tilde{p} := \sum_{n=0}^{\infty} p^{(n)}, \tag{1.9}$$

where $p_t^{(0)}(x,y) := p_t(x,y)$, and

$$p_t^{(n)}(x,y) := \int_0^t \int_X p_s(x,z)\, q(z) p_{t-s}^{(n-1)}(z,y)\, m(dz)\, ds, \quad n \geq 1. \tag{1.10}$$

Lemma 1.5. *$\tilde{p}$ is a transition density.*

This is indeed *similar* to Exercise 1.3. For details, see [15] and Lecture 2. Recall that h is supermedian for p. Here is a deeper result.

Theorem 1.6 ([11])**.** *We have $\tilde{p}_t h \leq h$ for all $t > 0$.*

Proof. For $n = 0, 1, \ldots$ and $t > 0$, $x \in X$, we consider

$$p_t^{(n)} h(x) := \int_X p_t^{(n)}(x,y) h(y)\, m(dy),$$

and we claim that

$$\sum_{k=0}^{n} p_t^{(k)} h(x) \leq h(x).$$

By (1.6) this holds for $n = 0$. By (1.10), Tonelli, induction and (1.7),

$$\begin{aligned}\sum_{k=1}^{n+1} p_t^{(k)} h(x) &= \int_0^t \int_X \int_X p_s(x,z)\, q(z) \sum_{k=0}^{n} p_{t-s}^{(k)}(z,y) h(y)\, m(dy)\, m(dz)\, ds \\ &\le \int_0^t \int_X p_s(x,z)\, q(z) h(z)\, m(dz)\, ds \\ &\le \int_0^t \int_X p_s(x,z) \int_0^\infty f'(u) \int_X p_u(z,w)\, \mu(dw)\, du\, m(dz)\, ds. \\ &= \int_0^t \int_0^\infty f'(u) p_{s+u}\mu(x)\, du\, ds,\end{aligned}$$

where in the last passage we used (1.2) and (1.4). By (1.3),

$$\begin{aligned}\sum_{k=1}^{n+1} p_t^{(k)} h(x) &\le \int_0^\infty \int_0^{u\wedge t} f'(u-s)\, ds\, p_u\mu(x)\, du \\ &\le \int_0^\infty [f(u) - f(u - u\wedge t)]\, p_u\mu(x)\, du \\ &= \int_0^\infty [f(u) - f(u-t)]\, p_u\mu(x)\, du.\end{aligned}$$

Now, for $k = 0$,

$$\begin{aligned}p_t h(x) &= \int_t^\infty f(u-t) p_u\mu(x)\, du \\ &= \int_0^\infty f(u-t)\, p_u\mu(x)\, du.\end{aligned}$$

So

$$\sum_{k=0}^{n+1} p_t^{(k)} h(x) \le \int_0^\infty f(u) p_u\mu(x)\, du = h(x).$$

The claim is proved. The theorem follows by letting $n \to \infty$. □

Remark 1.7. Theorem 1.6 asserts that h is supermedian for $\tilde{p}$. This is much more than (1.6), but (1.6) may also be useful in applications. Incidentally, the inequality in Theorem 1.6 gives an *integral finiteness* or *non-explosion* for the Schrödinger perturbation $\tilde{p}$, if $h(x) < \infty$.

In the next subsection, q will double as a weight in a Hardy inequality.

1.6. **Hardy inequality.** Let p, f, μ, h and q be as defined above.

Additionally, we shall assume that $\int_X p_t(x,y)m(dy) \le 1$ for all $t>0$ and $x \in X$. In short, p is a subprobability transition density. By Holmgren criterion [43, Theorem 3, p. 176], we then have $p_t u \in L^2(m)$ for each $u \in L^2(m)$, in fact $\int_X [p_t u(x)]^2 m(dx) \le \int_X u(x)^2 m(dx)$. Here $L^2(m)$ is the collection of all the real-valued square-integrable $\mathcal{M}$-measurable functions on X, equipped with the scalar product $\langle u,v\rangle := \int_X u(x)v(x)m(dx)$. Since the semigroup of operators $(p_t, t>0)$ is self-adjoint and *weakly measurable*,

$$\langle p_t u, u\rangle = \int_{[0,\infty)} e^{-\lambda t} d\langle P_\lambda u, u\rangle,$$

where P_λ is the *spectral decomposition* of the operators, see [39, Section 22.3]. For $u \in L^2(m)$ and $t>0$, we let

$$\mathcal{E}^{(t)}(u,u) := \frac{1}{t}\langle u - p_t u, u\rangle.$$

In the theory of Dirichlet forms, it is usually argued by the spectral theorem that $t \mapsto \mathcal{E}^{(t)}(u,u)$ is positive and decreasing [36, Lemma 1.3.4], allowing to define the quadratic form of p,

$$\mathcal{E}(u,u) := \lim_{t\to 0} \mathcal{E}^{(t)}(u,u), \quad u \in L^2(m). \tag{1.11}$$

Exercise 1.8. Check the *monotonicity.*

Hints. According to the spectral theorem, it is enough to verify that the function $[0,\infty) \ni t \mapsto (1-e^{-t\lambda})/t$ is decreasing for $\lambda \ge 0$, which follows by calculus.

Here is a Hardy inequality with a remainder (1.12) and a Hardy identity, or ground-state representation (1.13) of $\mathcal{E}$.

Theorem 1.9 ([11])**.** *If $u \in L^2(m)$ and $u=0$ on $\{x \in X : h(x) = 0$ or $\infty\}$,*

$$\begin{aligned}\mathcal{E}(u,u) \ge & \int_X u(x)^2 q(x)\, m(dx) \\ & + \liminf_{t\to 0} \int_X \int_X \frac{p_t(x,y)}{2t} \left(\frac{u(x)}{h(x)} - \frac{u(y)}{h(y)}\right)^2 h(y)h(x)m(dy)m(dx).\end{aligned} \tag{1.12}$$

If $f(t) = t_+^\beta$ with $\beta \ge 0$ in (1.5) or, more generally, if f is absolutely continuous and there are $\delta > 0$ and $c < \infty$ such that

$$[f(s) - f(s-t)]/t \le c f'(s) \qquad \text{for all } s>0 \text{ and } 0<t<\delta,$$

then for every $u \in L^2(m)$,

$$\mathcal{E}(u,u) = \int u(x)^2 q(x)\, m(dx) \tag{1.13}$$
$$+ \lim_{t\to 0} \int_X \int_X \frac{p_t(x,y)}{2t} \left(\frac{u(x)}{h(x)} - \frac{u(y)}{h(y)} \right)^2 h(y)h(x)m(dy)m(dx).$$

Proof. Let $v := u/h$, where $v(x) := 0$ if $h(x) = 0$ or ∞. Let $t > 0$. We note that $|vh| \le |u|$, thus $vh \in L^2(m)$ and by (1.6), $vp_th \in L^2(m)$. We then have

$$\mathcal{E}^{(t)}(vh, vh) = \langle v\frac{h - p_th}{t}, vh\rangle + \langle \frac{vp_th - p_t(vh)}{t}, vh\rangle =: I_t + J_t.$$

By the definition of J_t and the symmetry of p_t,

$$J_t = \frac{1}{t} \int_X \int_X p_t(x,y)[v(x) - v(y)]h(y)\, m(dy)\, v(x)h(x)\, m(dx)$$
$$= \int_X \int_X \frac{p_t(x,y)}{2t} [v(x) - v(y)]^2 h(x)h(y)\, m(dx)\, m(dy) \ge 0.$$

To deal with I_t, we let $x \in X$, assume that $h(x) < \infty$, and consider

$$(h - p_th)(x) = \int_0^\infty f(s)p_s\mu(x)\, ds - \int_0^\infty f(s)p_{s+t}\mu(x)\, ds$$
$$= \int_0^\infty [f(s) - f(s-t)]p_s\mu(x)\, ds.$$

Thus,

$$I_t = \int_X v^2(x)h(x) \int_0^\infty \frac{1}{t}[f(s) - f(s-t)]\; p_s\mu(x)\, ds\, m(dx).$$

By (1.11) and Fatou's lemma,

$$\mathcal{E}(vh, vh) \ge \int_X \int_0^\infty f'(s)p_s\mu(x)\, ds\, v^2(x)h(x)\, m(dx)$$
$$+ \liminf_{t\to 0} \int_X \int_X \frac{p_t(x,y)}{2t} [v(x) - v(y)]^2\, h(y)h(x)m(dy)m(dx)$$
$$= \int_X v^2(x)h^2(x)q(x)m(dx)$$
$$+ \liminf_{t\to 0} \int_X \int_X \frac{p_t(x,y)}{2t} [v(x) - v(y)]^2\, h(y)h(x)m(dy)m(dx).$$

Now we substitute u for vh. For remaining (minor) details, see [11]. □

Here is a resulting Hardy-type inequality.

Corollary 1.10. *For every* $u \in L^2(m)$ *we have* $\mathcal{E}(u,u) \geq \int_X u(x)^2 q(x)\, m(dx)$.

We are interested in quotients q as large as possible. This calls for explicit formulas or lower bounds of the numerator and upper bounds of the denominator. For instance, Exercise 1.2 yields the classical Hardy inequality:

Corollary 1.11. *The quadratic form of* $u \in L^2(\mathbb{R}^d, dx)$ *for the Gaussian semigroup is bounded below by* $(d/2-1)^2 \int_{\mathbb{R}^d} u(x)^2 |x|^{-2} dx$.

Below we discuss further applications. To this end we use the Fourier transform (in the version consistent with the characteristic function):

$$\hat{f}(\xi) := \int_{\mathbb{R}^d} e^{i\xi\cdot x} f(x) dx \quad \text{for (a.e.) } \xi \in \mathbb{R}^d,$$

where $\xi \cdot x := \xi_1 x_1 + \ldots + \xi_d x_d$. For instance,

$$\hat{g}_t(\xi) = e^{-t|\xi|^2}, \quad t > 0, \quad \xi \in \mathbb{R}^d.$$

According to Plancherel theorem, for $f, g \in L^2(dx)$,

$$\int_{\mathbb{R}^d} \hat{f}(\xi)\overline{\hat{g}(\xi)} d\xi = (2\pi)^d \int_{\mathbb{R}^d} f(x)\overline{g(x)} dx.$$

Exercise 1.12. Check this for $g_{1/2}$.

Hints. By inspection, $\hat{g}_{1/2} = (2\pi)^{d/2} g_{1/2}$, which agrees with Plancherel.

Exercise 1.13. The classical Hardy inequality in $\mathbb{R}^d$ may be stated as

$$\int_{\mathbb{R}^d} |\xi|^2 |\hat{u}(\xi)|^2 d\xi \geq \left(\frac{d-2}{2}\right)^2 (2\pi)^d \int_{\mathbb{R}^d} u(x)^2 |x|^{-2} dx, \quad d \geq 3.$$

Check this. Find a formulation that does not use the Fourier transform $\hat{u}$.

Hints. By Plancherel theorem, for $t > 0$ and $u \in L^2(\mathbb{R}^d)$,

$$\begin{aligned}\mathcal{E}^{(t)}(u,u) &:= \frac{1}{t}\langle u - p_t u, u\rangle \\ &= (2\pi)^d \int_{\mathbb{R}^d} \frac{1}{t}(1 - e^{-t|\xi|^2}) |\hat{u(\xi)}|^2 d\xi.\end{aligned}$$

Note Proposition 1.27 below, too.

1.7. **The isotropic α-stable semigroup.** A comprehensive reference is [42]. Let

$$\nu(z) := c_{d,\alpha}|z|^{-d-\alpha}, \quad z \in \mathbb{R}^d,$$

where $0 < \alpha < 2$, $d \in \mathbb{N}$, and the constant $c_{d,\alpha}$ is such that

$$\int_{\mathbb{R}^d} \big(1 - \cos(\xi \cdot z)\big)\nu(z)dz = |\xi|^\alpha, \quad \xi \in \mathbb{R}^d.$$

Note that the measure $\nu(z)dz$ satisfies the so-called Lévy-measure condition:

$$\int_{\mathbb{R}^d} (1 \wedge |x|^2)\nu(z)dz < \infty.$$

Further, it is homogeneous of degree $-\alpha$: $\int_{kA} \nu(z)dz = k^{-\alpha}\int_A \nu(z)dz$, $k > 0$, $A \subset \mathbb{R}^d$, and it is invariant upon (linear) unitary transformations $T : \mathbb{R}^d \to \mathbb{R}^d$ (to wit, $T^*T = TT^* = I$) because $\nu(Tz) = \nu(z)$.

Exercise 1.14. Prove that, indeed, for some $c \in (0, \infty)$,

$$\int_{\mathbb{R}^d} \big(1 - \cos(\xi \cdot z)\big)|z|^{-d-\alpha}dz = c|\xi|^\alpha, \quad \xi \in \mathbb{R}^d.$$

Hints. The left hand side is invariant upon (linear) unitary transformations $T : \mathbb{R}^d \to \mathbb{R}^d$ (to wit, $T^*T = TT^* = I$). For $\xi \neq 0$, consider $\xi = |\xi||\xi|^{-1}\xi$ and change the variables $z := |\xi|y$. Finally, use polar coordinates and note that $0 \le 1 - \cos s = \int_0^s \sin r dr \le \frac{s^2}{2} \wedge 2$ for $s \in \mathbb{R}$.

Remark 1.15. It is known that $c_{d,\alpha} = 2^\alpha\Gamma\big((d+\alpha)/2\big)\pi^{-d/2}/|\Gamma(-\alpha/2)|$.

For $t > 0$, we let

$$p_t(x) := (2\pi)^{-d}\int_{\mathbb{R}^d} e^{-t|\xi|^\alpha}e^{-i\xi\cdot x}d\xi, \quad x \in \mathbb{R}^d.$$

By the celebrated Lévy-Khintchine formula, p_t is a probability density and

$$\hat{p_t}(\xi) := \int_{\mathbb{R}^d} e^{i\xi\cdot x}\, p_t(x)dx = e^{-t|\xi|^\alpha}, \quad \xi \in \mathbb{R}^d,\ t > 0.$$

For $\alpha = 1$, we get the Cauchy convolution semigroup (aka Poisson kernel in Harmonic Analysis):

$$p_t(z) = \Gamma((d+1)/2)\pi^{-(d+1)/2}\frac{t}{\big(|z|^2 + t^2\big)^{(d+1)/2}}\,.$$

Exercise 1.16. Prove that for every $\alpha \in (0,2)$,

$$p_t(z) = t^{-d/\alpha} p_1(t^{-1/\alpha} z)\,, \quad t > 0\,,\ z \in \mathbb{R}^d\,.$$

Hints. Use the homogeneity of $|\xi|^\alpha$ and uniqueness for Fourier transform.

Remark 1.17. It is known that $p_t(x)/t \to \nu(x)$ for $x \in \mathbb{R}^d$ as $t \to 0$.

Exercise 1.18. Check this directly for $\alpha = 1$.

Hints. Do we get $c_{d,1}$?

Apart from obvious similarities, there exist important differences between p (hence $0 < \alpha < 2$) and g (hence $\alpha = 2$). For instance the decay of p in space is polynomial (see, e.g., [22] for a proof):

Lemma 1.19. *There exists $c = c(d,\alpha)$ such that, for all $z \in \mathbb{R}^d$, $t > 0$,*

$$c^{-1}\left(\frac{t}{|z|^{d+\alpha}} \wedge t^{-d/\alpha}\right) \le p_t(z) \le c\left(\frac{t}{|z|^{d+\alpha}} \wedge t^{-d/\alpha}\right)\,.$$

1.8. **Subordination.** There is a convolution semigroup η_t, $t > 0$, of probability densities concentrated on $(0,\infty)$, that is, such that $\eta_t(s) = 0$, $s \le 0$ and $\eta_r * \eta_t = \eta_{r+t}$ for $r, t > 0$, which satisfy

$$\int_0^\infty e^{-us}\eta_t(s)\,ds = e^{-tu^{\alpha/2}}, \quad u \ge 0. \tag{1.14}$$

We have, using *Bochner subordination*,

$$p_t(x) := \int_0^\infty g_s(x)\eta_t(s)\,ds,$$

where g is the Gaussian kernel defined in (1.1). This is great to analyze p_t.

Exercise 1.20. Find $\hat{p}_t$ using (1.14).

Hints. For $t > 0$,

$$p_t(x) := \int_0^\infty g_s(x)\eta_t(s)\,ds,$$

so

$$\hat{p}_t(\xi) = \int_0^\infty e^{-s|\xi|^2}\eta_t(s)\,ds, \quad \xi \in \mathbb{R}^d.$$

The result follows by (1.14) and the definition of the gamma function.

Below we denote

$$\nu(x,y) := \nu(y-x)$$

and

$$p_t(x,y) := p_t(y-x).$$

Exercise 1.21. Verify that

$$\mathcal{E}^{(t)}(u,u) = \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} [u(x)-u(y)]^2 \, \frac{1}{t} p_t(x,y) dx dy.$$

1.9. **Fractional Hardy inequality.** Regarding the setting of Subsection 1.6, we will have $m(dx) = dx$, the Lebesgue measure on $\mathbb{R}^d$. For $u \in L^2(\mathbb{R}^d, dx)$, we let

$$\mathcal{E}(u,u) := \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} [u(x)-u(y)]^2 \nu(x,y)\, dy\, dx. \tag{1.15}$$

The following statement on *self-dominated convergence* is quite useful.

Lemma 1.22. [18, Lemma 6] *If $f, f_k : \mathbb{R}^d \to [0,\infty]$ satisfy $f_k \le cf$ and $f = \lim_{k\to\infty} f_k$, $k = 1, 2, \ldots$, then for each measure μ, $\lim_{k\to\infty} \int f_k \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu$.*

Proof. The integrals converge either by the dominated convergence theorem, if the right hand side is finite, or – else – by Fatou's lemma. □

Exercise 1.23. Prove that (1.15) is the Dirichlet form of p.

Hints. The result follows from Lemma 1.22, Remark 1.17, and Lemma 1.19.

The important case $\beta = (d-\alpha)/(2\alpha)$ in the following Hardy equality for the Dirichlet form of the fractional Laplacian was given by Frank, Lieb and Seiringer in [34, Proposition 4.1] (see [5] for another proof; see also [38]). In fact, [34, formula (4.3)] also covers the case of $(d-\alpha)/(2\alpha) \le \beta \le (d-\alpha)/\alpha$ and smooth compactly supported functions u in the following Proposition. Our approach is different from that of [34, Proposition 4.1] because we do not use the Fourier transform.

Proposition 1.24 ([11]). *If $0 < \alpha < d$, $0 < \beta < (d-\alpha)/\alpha$, $u \in L^2(\mathbb{R}^d)$, then*

$$\mathcal{E}(u,u) = C\int_{\mathbb{R}^d} \frac{u(x)^2}{|x|^\alpha}\, dx + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left(\frac{u(x)}{h(x)} - \frac{u(y)}{h(y)}\right)^2 h(x)h(y)\nu(x,y)\, dy\, dx\,,$$

where $h(x) = |x|^{\alpha(\beta+1)-d}$ *and*

$$C = 2^\alpha \Gamma(\frac{d}{2} - \tfrac{\alpha\beta}{2})\Gamma(\frac{\alpha(\beta+1)}{2})\Gamma(\frac{d}{2} - \tfrac{\alpha(\beta+1)}{2})^{-1}\Gamma(\frac{\alpha\beta}{2})^{-1}.$$

We get a maximal $C = 2^\alpha \Gamma(\frac{d+\alpha}{4})^2/\Gamma(\frac{d-\alpha}{4})^2$ *if* $\beta = (d-\alpha)/(2\alpha)$.

Exercise 1.25. Prove this ground-state representation using Theorem 1.9.

Hints. Let $-1 < \beta < d/\alpha - 1$. For $f(t) := t_+^\beta$ and σ-finite Borel measure $\mu \geq 0$ on $\mathbb{R}^d$ we have

$$\begin{aligned} h(x) &:= \int_0^\infty \int_{\mathbb{R}^d} f(t) p_t(x-y)\mu(dy)\,dt \\ &= \int_0^\infty \int_{\mathbb{R}^d} t^\beta \int_0^\infty g_s(x-y)\eta_t(s)\,ds\,\mu(dy)\,dt \\ &= \int_{\mathbb{R}^d} \int_0^\infty \int_0^\infty t^\beta \eta_t(s)\,dt\,g_s(x-y)\,ds\,\mu(dy) \\ &= \int_{\mathbb{R}^d} \int_0^\infty \frac{\Gamma(\beta+1)}{\Gamma(\frac{\alpha(\beta+1)}{2})} s^{\frac{\alpha(\beta+1)}{2}-1} g_s(x-y)\,ds\,\mu(dy) \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\frac{\alpha(\beta+1)}{2})} \frac{\Gamma(\frac{d}{2} - \frac{\alpha(\beta+1)}{2})}{4^{\frac{\alpha(\beta+1)}{2}}\pi^{d/2}} \int_{\mathbb{R}^d} |x-y|^{\alpha(\beta+1)-d}\,\mu(dy), \end{aligned}$$

where in the last two equalities we assume $\alpha(\beta+1)/2 - 1 < d/2 - 1$ and use (1.8). If, furthermore, $\beta \geq 0$, then by the same calculation

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}^d} f'(t) p_t(x,y)\mu(dy)\,dt \\ &\qquad = \beta \frac{\Gamma(\beta)}{\Gamma(\frac{\alpha\beta}{2})} 4^{-\frac{\alpha\beta}{2}} \pi^{-d/2} \Gamma(\frac{d}{2} - \tfrac{\alpha\beta}{2}) \int_{\mathbb{R}^d} |x-y|^{\alpha\beta-d}\,\mu(dy). \end{aligned}$$

Here the expression is zero if $\beta = 0$. If $\mu = \delta_0$, then we get

$$h(x) = \frac{\Gamma(\beta+1)}{\Gamma(\frac{\alpha(\beta+1)}{2})} \frac{\Gamma(\frac{d}{2} - \frac{\alpha(\beta+1)}{2})}{4^{\frac{\alpha(\beta+1)}{2}}\pi^{d/2}} |x|^{\alpha(\beta+1)-d}$$

and

$$\begin{aligned} q(x) &= \frac{1}{h(x)} \int_0^\infty \int_{\mathbb{R}^d} f'(t) p_t(x,y)\mu(dy)\,dt \\ &= \frac{4^{\alpha/2}\Gamma(\frac{d}{2} - \frac{\alpha\beta}{2})\Gamma(\frac{\alpha(\beta+1)}{2})}{\Gamma(\frac{d}{2} - \frac{\alpha(\beta+1)}{2})\Gamma(\frac{\alpha\beta}{2})} |x|^{-\alpha}. \end{aligned}$$

By homogeneity, we may assume $h(x) = |x|^{\alpha(\beta+1)-d}$, without changing q. By the second statement of Theorem 1.9, it remains to show that

$$\lim_{t\to 0}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{p_t(x,y)}{2t}\left[\frac{u(x)}{h(x)}-\frac{u(y)}{h(y)}\right]^2 h(y)h(x)dydx$$
$$= \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left[\frac{u(x)}{h(x)}-\frac{u(y)}{h(y)}\right]^2 h(y)h(x)\nu(x,y)\,dy\,dx\,. \tag{1.16}$$

Since $p_t(x,y)/t \le c\nu(x,y)$ [14], and $p_t(x,y)/t \to \nu(x,y)$ as $t \to 0$, (1.16) follows by Lemma 1.22. If $\alpha\beta = (d-\alpha)/2$, then we obtain $h(x) = |x|^{-(d-\alpha)/2}$ and

$$q(x) = \frac{4^{\alpha/2}\Gamma(\frac{d+\alpha}{4})^2}{\Gamma(\frac{d-\alpha}{4})^2}|x|^{-\alpha}.$$

Finally, the statement of the proposition is trivial for $\beta = d/\alpha - 1$.

Corollary 1.26 ([11])**.** *If $0 \le r \le d-\alpha$, $x \in \mathbb{R}^d$ and $t > 0$, then*

$$\int_{\mathbb{R}^d} p_t(y-x)|y|^{-r}dy \le |x|^{-r}.$$

If $0 < r < d-\alpha$, $x \in \mathbb{R}^d$, $t > 0$, $\beta = (d-\alpha-r)/\alpha$, and $\tilde{p}$ is given by (1.9),

$$\int_{\mathbb{R}^d} \tilde{p}_t(y-x)|y|^{-r}dy \le |x|^{-r}.$$

Proof. By (1.6) and the proof of Proposition 1.24, we get the first estimate. The second estimate is stronger because $\tilde{p} \ge p$, and it follows from Theorem 1.6, see the proof of Proposition 1.24. □

1.10. **Further information about the classical Hardy identity.** For completeness we state Hardy identities for the Dirichlet form of the Gaussian semigroup on $\mathbb{R}^d$. Namely, (1.18) below is the optimal classical Hardy equality with remainder, and (1.17) is its slight extension, in the spirit of Proposition 1.24. For the equality (1.18), see for example [32, formula (2.3)], [35, Section 2.3] or [5]. Equality (1.17) may also be considered as a corollary of [35, Section 2.3]. Note the restriction of domain, compared to Corollary 1.11.

Proposition 1.27. *Suppose $d \ge 3$ and $0 \le \gamma \le d-2$. For $u \in W^{1,2}(\mathbb{R}^d)$,*

$$\int_{\mathbb{R}^d}|\nabla u(x)|^2dx = \gamma(d-2-\gamma)\int_{\mathbb{R}^d}\frac{u(x)^2}{|x|^2}dx + \int_{\mathbb{R}^d}\left|h(x)\nabla\frac{u}{h}(x)\right|^2 dx, \tag{1.17}$$

where $h(x) = |x|^{\gamma+2-d}$. In particular,

$$\int_{\mathbb{R}^d} |\nabla u(x)|^2\,dx = \frac{(d-2)^2}{4}\int_{\mathbb{R}^d} \frac{u(x)^2}{|x|^2}\,dx + \int_{\mathbb{R}^d} \left| |x|^{\frac{2-d}{2}} \nabla \frac{u(x)}{|x|^{(2-d)/2}} \right|^2 dx. \tag{1.18}$$

The result has some ad-hoc elements (like gradient, ∇), so we refer to [11].

2. Schrödinger perturbations and...

The plan of this Lecture 2 is to discuss details of Schrödinger perturbations from [15], results on nonlocal Schrödinger perturbations from [23], and nonlocal boundary conditions in [20]. It would also be nice to mention gradient perturbation [16], general Schrödinger perturbations [19], special considerations for the Gaussian kernel [24], [9], [12], and critical Hardy-type Schrödinger perturbations [13]... Let us first make a probability connection.

2.1. **A Feynman-Kac formula (down with time-homogeneous notation!)** Here we follow [15]. Let $g(s,x,t,y) := g_{t-s}(y-x)$ be the Gaussian kernel in $\mathbb{R}^d$, $s,t \in \mathbb{R}$, $x,y \in \mathbb{R}^d$. (We let $g = 0$ if $s \geq t$.) Let $q : \mathbb{R} \times \mathbb{R}^d \to [0,\infty]$ (or $\mathbb{C}$). Here is the perturbation of g by q on $X = \mathbb{R}^d$ without the time-homogeneous corset: Let $\tilde{g} := \sum_{n=0}^{\infty} g^{(n)}$, where $g^{(0)}(s,x,t,y) := g(s,x,t,y)$, and for $n \geq 1$,

$$g^{(n)}(s,x,t,y) := \int_s^t \int_X g(s,x,u,z)\, q(z,u) g^{(n-1)}(u,z,t,y)\, m(dz)\, du.$$

Let $\mathbb{E}_{s,x}$ and $\mathbb{P}_{s,x}$ be the expectation and the distribution of the Brownian motion Y (here $Y_t = B_{2t}$) starting at the point $x \in \mathbb{R}^d$ at time $s \in \mathbb{R}$. So,

$$\mathbb{P}_{s,x}[Y_t \in A] = \int_A g(s,x,t,y)\,dy, \quad t > s,\ A \subset \mathbb{R}^d.$$

Y has transition probability density $g(u_1, z_1, u_2, z_2)$, where $s \leq u_1 < u_2$. Thus, the finite dimensional distributions have the density functions

$$g(s,x,u_1,z_1)g(u_1,z_1,u_2,z_2)\cdots g(u_{n-1},z_{n-1},u_n,z_n)\,.$$

Further, for $y \in \mathbb{R}^d$, $t > s$, we let $\mathbb{E}^{t,y}_{s,x}$ and $\mathbb{P}^{t,y}_{s,x}$ denote the expectation and the distribution of the process starting at x at time s and conditioned to reach

y at time t (Brownian bridge). The bridge, also denoted Y, has transition probability density

$$r(u_1, z_1, u_2, z_2) = \frac{g(u_1, z_1, u_2, z_2) g(u_2, z_2, t, y)}{g(u_1, z_1, t, y)},$$

where $s \le u_1 < u_2 < t$ and $z_1, z_2 \in \mathbb{R}^d$. Thus, its finite dimensional distributions have the density functions

$$\frac{g(s, x, u_1, z_1) g(u_1, z_1, u_2, z_2) \cdots g(u_n, z_n, t, y)}{g(s, x, t, y)}. \tag{2.1}$$

Here $s \le u_1 < \ldots < u_n < t$, $z_1, \ldots, z_n \in \mathbb{R}^d$. We get a *disintegration* of $\mathbb{P}_{s,x}$:

$$\begin{aligned}
&\mathbb{P}_{s,x}\left(Y_{u_1} \in A_1, \ldots, Y_{u_n} \in A_n, Y_t \in B\right) \\
&= \int_B \mathbb{P}^{t,y}_{s,x}\left(Y_{u_1} \in A_1, \ldots, Y_{u_n} \in A_n\right) g(s, x, t, y)\, dy, \; A_1, \ldots, A_n, B \subset \mathbb{R}^d.
\end{aligned}$$

Here comes the *multiplicative functional* $e_q(s,t) := \exp\left(\int_s^t q(u, Y_u)\, du\right)$ [27]. Of course,

$$\mathbb{E}^{t,y}_{s,x}\, e_q(s,t) = \sum_{n=0}^{\infty} \frac{1}{n!}\, \mathbb{E}^{t,y}_{s,x} \left(\int_s^t q(u, Y_u)\, du\right)^n.$$

According to (2.1),

$$\begin{aligned}
\mathbb{E}^{t,y}_{s,x} \int_s^t q(u, Y_u)\, du &= \int_s^t \int_{\mathbb{R}^d} \frac{g(s, x, u, z) q(u, z) g(u, z, t, y)}{g(s, x, t, y)}\, du dz \\
&= \frac{g_1(s, x, t, y)}{g(s, x, t, y)}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\mathbb{E}^{t,y}_{s,x} \frac{1}{2} \left(\int_s^t q(u, Y_u)\, du\right)^2 = \mathbb{E}^{t,y}_{s,x} \int_s^t \int_u^t q(u, Y_u) q(v, Y_v)\, dv du \\
= &\int_s^t \int_u^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{g(s, x, u, z) g(u, z, v, w) g(v, w, t, y)}{g(s, x, t, y)} q(u, z) q(v, w)\, dw dz\, dv du \\
= &\int_s^t \int_{\mathbb{R}^d} \frac{g(s, x, u, z) g_1(u, z, t, y)}{g(s, x, t, y)} q(u, z)\, dz\, du = \frac{g_2(s, x, t, y)}{g(s, x, t, y)}.
\end{aligned}$$

Similarly, for every $n = 0, 1, \ldots$,

$$\frac{1}{n!}\,\mathbb{E}_{s,x}^{t,y}\left(\int_s^t q(u, Y_u)\,du\right)^n = \frac{g_n(s,x,t,y)}{g(s,x,t,y)}\,,$$

hence we get a Feynmann-Kac formula

$$\tilde{g}(s,x,t,y) = g(s,x,t,y)\,\mathbb{E}_{s,x}^{t,y}\exp\int_s^t q(u, Y_u)du\,.$$

We may interpret $\tilde{g}(s,x,t,y)/g(s,x,t,y)$ as the eventual inflation of mass of the Brownian particle moving from (s,x) to (t,y). The mass grows *multiplicatively* where $q > 0$ (and decreases if $q < 0$). For instance, if $q(u,z) = q(u)$ (depends only on time), then

$$\tilde{g}(s,x,t,y)/g(s,x,t,y) = \exp\left(\int_s^t q(u)du\right).$$

2.2. **Integral kernels.** Here we mostly follow [23]. Let $(E, \mathcal{E})$ be a measurable space. A kernel on E is a map K from $E \times \mathcal{E}$ to $[0, \infty]$ such that

$$x \mapsto K(x, A) \text{ is } \mathcal{E}\text{-measurable for all } A \in \mathcal{E}, \text{ and}$$

$$A \mapsto K(x, A) \text{ is countably additive for all } x \in E.$$

Consider kernels K and J on E. The map $(E \times \mathcal{E}) \to [0, \infty]$ given by

$$(x, A) \mapsto \int_E K(x, dy) J(y, A)$$

is another kernel on E, called the *composition* of K and J, and denoted KJ.

Exercise 2.1. Why is composition of kernels similar to multiplication of matrices?

Hints. In fact it is similar to multiplication of positive square matrices, which is a special case of composition of kernels.

We let $K_n := K_{n-1}JK(s, x, A) = (KJ)^n K$, $n = 0, 1, \ldots$. The composition of kernels is associative, which yields the following lemma.

Lemma 2.2. *$K_n = K_{n-1-m}JK_m$ for all $n \in \mathbb{N}$ and $m = 0, 1, \ldots, n-1$.*

We define the *perturbation*, $\widetilde{K}$, of K by J, via the *perturbation series*,

$$\widetilde{K} := \sum_{n=0}^{\infty} K_n = \sum_{n=0}^{\infty} (KJ)^n K. \tag{2.2}$$

Of course, $K \le \widetilde{K}$, and we have the following *perturbation formula(s)*,

$$\widetilde{K} = K + \widetilde{K}JK = K + KJ\widetilde{K}. \tag{2.3}$$

Goals: *algebra* or *bounds* for $\widetilde{K}$ under additional conditions on K and J.

2.3. **An upper bound.** Consider a set X (the space) with σ-algebra $\mathcal{M}$, the real line $\mathbb{R}$ (the time) with the Borel sets $\mathcal{B}_{\mathbb{R}}$, and the space-time,

$$E := \mathbb{R} \times X,$$

with the product σ-algebra $\mathcal{E} = \mathcal{B}_{\mathbb{R}} \times \mathcal{M}$. Let $\eta \in [0, \infty)$ and a function $Q : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ satisfy the following condition of *super-additivity*:

$$Q(u, r) + Q(r, v) \le Q(u, v) \quad \text{for all } u < r < v.$$

Exercise 2.3. Check $Q(s,t) := \int_s^t f(u)du$ is superadditive if $f : \mathbb{R} \to [0,\infty)$.

Hints. In fact, this Q is additive:-)

Let J be another kernel on E. We assume that K and J are *forward* kernels, i.e., for $A \in \mathcal{E}$, $s \in \mathbb{R}$, $x \in X$,

$$K(s,x,A) = 0 = J(s,x,A) \text{ whenever } A \subseteq (-\infty, s] \times X.$$

It also *suffices* that K is forward and J is *instantaneous*, that is, $J(s,x,dtdy) = j(s,x,dy)\delta_s(dt)$. In particular, Schrödinger perturbations are obtained when $j(s,x,dy) = q(s,x)\delta_x(dy)$ is *local*. In what follows, we study consequences of the following assumption,

$$K_1(s,x,A) := KJK(s,x,A) \leq \int_A [\eta + Q(s,t)]K(s,x,dtdy), \tag{2.4}$$

with *impulsive* bound $\eta \in [0,\infty)$ and *superadditive* bound Q.

Theorem 2.4. *Assuming* (2.4)*, for all* $n = 1,2,\ldots,$ *and* $(s,x) \in E$*, we have*

$$\begin{aligned} K_n(s,x,dtdy) &\leq K_{n-1}(s,x,dtdy)\left[\eta + \frac{Q(s,t)}{n}\right] \\ &\leq K(s,x,dtdy)\prod_{l=1}^{n}\left[\eta + \frac{Q(s,t)}{l}\right]. \end{aligned}$$

If $0 < \eta < 1$*, then for all* $(s,x) \in E$*,*

$$\widetilde{K}(s,x,dtdy) \leq K(s,x,dtdy)\left(\frac{1}{1-\eta}\right)^{1+Q(s,t)/\eta}.$$

If $\eta = 0$*, then for all* $(s,x) \in E$*,*

$$\widetilde{K}(s,x,dtdy) \leq K(s,x,dtdy)e^{Q(s,t)}.$$

2.4. **Pointwise versions.** Theorem 2.4 has two *pointwise* variants (which may be skipped). Fix a (nonnegative) σ-finite, non-atomic measure

$$dt := \mu(dt)$$

on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ and a function $k(s,x,t,A) \geq 0$ defined for $s,t \in \mathbb{R}$, $x \in X$, $A \in \mathcal{M}$, such that $k(s,x,t,dy)dt$ is a forward kernel and $(s,x) \mapsto k(s,x,t,A)$ is jointly measurable for all $t \in \mathbb{R}$ and $A \in \mathcal{M}$. Let $k_0 = k$, and for $n = 1, 2, \ldots$,

$$k_n(s,x,t,A) = \int_s^t \int_X k_{n-1}(s,x,u,dz) \int_{(u,t)\times X} J(u,z,du_1dz_1)k(u_1,z_1,t,A)du.$$

The perturbation, $\widetilde{k}$, of k by J, is defined as $\widetilde{k} = \sum_{n=0}^{\infty} k_n$. Assume that

$$\int_s^t \int_X k(s,x,u,dz) \int_{(u,t)\times X} J(u,z,du_1dz_1)k(u_1,z_1,t,A)du \leq [\eta + Q(s,t)]k(s,x,t,A).$$

Theorem 2.5. *Under the assumptions, for all $n = 1, 2, \ldots$, and $(s,x) \in E$,*

$$\begin{aligned} k_n(s,x,t,dy) &\leq k_{n-1}(s,x,t,dy)\left[\eta + \frac{Q(s,t)}{n}\right] \\ &\leq k(s,x,t,dy)\prod_{l=1}^{n}\left[\eta + \frac{Q(s,t)}{l}\right]. \end{aligned}$$

If $0 < \eta < 1$, then for all $(s,x) \in E$ and $t \in \mathbb{R}$ we have

$$\widetilde{k}(s,x,t,dy) \leq k(s,x,t,dy)\left(\frac{1}{1-\eta}\right)^{1+Q(s,t)/\eta}.$$

If $\eta = 0$, then

$$\widetilde{k}(s,x,t,dy) \leq k(s,x,t,dy)e^{Q(s,t)}.$$

For the *finest* variant of Theorem 2.4, we fix a σ-finite measure

$$dz := m(dz)$$

on $(X, \mathcal{M})$. We consider function $\kappa(s,x,t,y) \geq 0$, $s,t \in \mathbb{R}$, $x,y \in X$, such that $\kappa(s,x,t,y)dtdy$ is a forward kernel and $(s,x) \mapsto k(s,x,t,y)$ is jointly measurable for all $t \in \mathbb{R}$ and $y \in X$. We call such κ a (forward) *kernel density* (see [19]). We define $\kappa_0(s,x,t,y) = \kappa(s,x,t,y)$, and

$$\kappa_n(s,x,t,y) = \int_s^t \int_X \kappa_{n-1}(s,x,u,z) \int_{(u,t)\times X} J(u,z,du_1dz_1)\kappa(u_1,z_1,t,y)\,dz\,du\,,$$

where $n = 1, 2, \ldots$. Let $\widetilde{\kappa} = \sum_{n=0}^{\infty} \kappa_n$. For all $s < t \in \mathbb{R}$, $x, y \in X$, we assume

$$\int_s^t \int_X \kappa(s,x,u,z) \int_{(u,t)\times X} J(u,z,du_1dz_1)\kappa(u_1,z_1,t,y)dzdu \leq [\eta + Q(s,t)]\kappa(s,x,t,y).$$

Theorem 2.6. *Under the assumptions, for $n = 1, 2, \ldots$, $s < t$ and $x, y \in X$,*

$$\begin{aligned}\kappa_n(s,x,t,y) &\leq \kappa_{n-1}(s,x,t,y)\left[\eta + \frac{Q(s,t)}{n}\right] \\ &\leq \kappa(s,x,t,y)\prod_{l=1}^{n}\left[\eta + \frac{Q(s,t)}{l}\right].\end{aligned}$$

If $0 < \eta < 1$, then for all $s, t \in \mathbb{R}$ and $x, y \in X$,

$$\widetilde{\kappa}(s,x,t,y) \leq \kappa(s,x,t,y)\left(\frac{1}{1-\eta}\right)^{1+Q(s,t)/\eta}.$$

If $\eta = 0$, then

$$\widetilde{\kappa}(s,x,t,y) \leq \kappa(s,x,t,y)e^{Q(s,t)}.$$

Exercise 2.7. If $\kappa_1 \leq \eta\kappa$ with $\eta \in (0,1)$, then $\widetilde{\kappa} \leq \frac{1}{1-\eta}\kappa$ (Khasminski's lemma). Explain why this follows from the above. Also, verify it directly using perturbation series.

Hints. By induction, $\kappa_n \leq \eta^n \kappa$.

2.5. **Transition kernels.** Let k as above be a *transition kernel*, i.e., additionally satisfy the Chapman-Kolmogorov conditions for $s < u < t$, $A \in \mathcal{M}$ (we do *not* assume $k(s,x,t,X) = 1$),

$$\int_X k(s,x,u,dz)k(u,z,t,A) = k(s,x,t,A).$$

Following [15], we may show that $\widetilde{k}$ is a transition kernel, too. Here is the first step.

Lemma 2.8. *For all $s < u < t$, $x, y \in X$, $A \in \mathcal{M}$, and $n = 0, 1, \ldots$,*

$$\sum_{m=0}^{n}\int_X k_m(s,x,u,dz)k_{n-m}(u,z,t,A) = k_n(s,x,t,A). \tag{2.5}$$

Proof. We note that (2.5) is true for $n = 0$ by fact that k is a transition kernel and satisfies the Chapman-Kolmogorov equation. Assume that $n \geq 1$ and (2.5) holds for $n-1$. The sum of the first n terms on the left of (2.5) can be dealt with by induction:

$$\begin{aligned}
(2.6)\qquad & \sum_{m=0}^{n-1} \int_X k_m(s,x,u,dz) k_{n-m}(u,z,t,A) \\
&= \sum_{m=0}^{n-1} \int_X k_m(s,x,u,dz) \int_u^t \int_X k_{n-m-1}(u,z,r,dw) \\
&\quad \times \int_{(r,\infty)\times X} J(r,w,dr_1dw_1) k(r_1,w_1,t,A) dr \\
&= \int_u^t \int_X \int_{(r,\infty)\times X} J(r,w,dr_1dw_1) k(r_1,w_1,t,A) \\
&\quad \times \sum_{m=0}^{n-1} \int_X k_m(s,x,u,dz) k_{(n-1)-m}(u,z,r,dw) dr \\
&= \int_u^t \int_X k_{n-1}(s,x,r,dw) \int_{(r,\infty)\times X} J(r,w,dr_1dw_1) k(r_1,w_1,t,A) dr.
\end{aligned}$$

The $(n+1)$-st term on the left of (2.5) is

$$\begin{aligned}
& \int_X k_n(s,x,u,dz) k(u,z,t,A) \\
&= \int_X \int_s^u \int_X k_{n-1}(s,x,r,dw) \int_{(r,\infty)\times X} J(r,w,dr_1dw_1) k(r_1,w_1,u,dz) k(u,z,t,A) dr \\
&= \int_s^u \int_X k_{n-1}(s,x,r,dw) \int_{(r,\infty)\times X} J(r,w,dr_1dw_1) k(r_1,w_1,t,A) dr,
\end{aligned}$$

and (2.5) follows on this and (2.6). □

Here is the a consequence, the *evolution* (Chapman-Kolmogorov) property.

Lemma 2.9 (Chapman-Kolmogorov)**.** *For all $s < u < t$, $x, y \in \mathbb{R}^d$ and $A \in \mathcal{M}$,*
$$\int_X \widetilde{k}(s,x,u,dz)\widetilde{k}(u,z,t,A) = \widetilde{k}(s,x,t,A).$$

The proof follows that of [15, Lemma 2], using (2.5). Thus, $\widetilde{k}$ is a transition kernel. Similarly, $\widetilde{\kappa}$ above is a transition density, provided so is κ.

Exercise 2.10. Prove Lemma 2.9 in analogy to Exercise 1.3.

Remark 2.11. Estimating $K_1 := KJK$ by K is crucial. Much of our research (K.B., K.S. et al) was devoted to this goal, including proving and applying 3G Theorems for power-like kernels and 4G (4.5G) Theorems for others. See [19, 24, 9, 12]. See [13] for cases when we get $\widetilde{K}$ much bigger than K or even *explosion*; see [16] for gradient perturbations and [18, 17] for applications.

Remark 2.12. The *parametrix method* a related but more difficult subject, where we do not have an initial transition kernel to start with, but a *field* of transition kernels, see [25] and [45].

We can describe connections with 'generators'. For instance, let $p(s,x,t,y) := p_{t-s}(y-x)$ be the transition kernel of the α-stable semigroup, aka fundamental solution of $\partial_t - \Delta_y^{\alpha/2}$:

$$\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}^d} p(s,x,t,y)\left[\partial_t + \Delta_y^{\alpha/2}\right]\phi(t,y)\,dydt = -\phi(s,x)\,, \tag{2.7}$$

where $s \in \mathbb{R}$, $x \in \mathbb{R}^d$, and $\phi \in C_c^\infty(\mathbb{R}\times\mathbb{R}^d)$. (Hint: Use the Fourier transform on $\mathbb{R}^d$.)

Here $C_c^\infty(\mathbb{R}\times\mathbb{R}^d)$ is the class of all infinitely differentiable compactly supported functions on $\mathbb{R}\times\mathbb{R}^d$, and

$$\begin{aligned}\Delta^{\alpha/2}\phi(y) \;&:=\; -(-\Delta)^{\alpha/2}\phi(y) = \lim_{t\downarrow 0}\frac{p_t\phi(y)-\phi(y)}{t}\\ &=\; \frac{2^\alpha\Gamma((d+\alpha)/2)}{\pi^{d/2}|\Gamma(-\alpha/2)|}\lim_{\varepsilon\downarrow 0}\int_{\{|z|>\varepsilon\}}\frac{\phi(y+z)-\phi(y)}{|z|^{d+\alpha}}dz\,, \quad y\in\mathbb{R}^d\,.\end{aligned}$$

Let $\big(L\phi\big)(t,y) = \partial_t\phi(t,y) + \Delta_y^{\alpha/2}\phi(t,y)$, the parabolic operator.

We also consider kernels $Q(s,x,dudz) := q(s,x)\delta_s(du)\delta_x(dz)$, the kernel of multiplication by q, and $P(s,x,dudz) := p(s,x,u,z)dudz$, and

$$\tilde{P} := \sum_{n=0}^{\infty} (PQ)^n P\,.$$

We can interpret the fundamental solution (2.7) as

$$PL\phi = -\phi \qquad (\phi \in C_c^\infty(\mathbb{R}\times\mathbb{R}^d))\,. \tag{2.8}$$

Let us assume, e.g., that $Q \geq 0$ and $PQP \leq \eta P$ for some $\eta \in [0,1)$. Then

$$\tilde{P}(L+Q)\phi = -\phi \qquad (\phi \in C_c^\infty(\mathbb{R}\times\mathbb{R}^d))\,. \tag{2.9}$$

Indeed, by (2.8),

$$\begin{aligned} \tilde{P}(L+Q)\phi &= \sum_{n=0}^{\infty} P(QP)^n(L+Q)\phi \\ &= PL\phi + \sum_{n=1}^{\infty}(PQ)^n PL\phi + \sum_{n=0}^{\infty}(PQ)^{n+1}\phi = -\phi\,. \end{aligned}$$

Here is what (2.9) means:

$$\int_{\mathbb{R}}\int_{\mathbb{R}^d} \tilde{p}(s,x,t,y)\big[\partial_t\phi(t,y) + \Delta_y^{\alpha/2}\phi(t,y) + q(t,y)\phi(t,y)\big]\,dydt = -\phi(s,x)\,,$$

where $s\in\mathbb{R}$, $x\in\mathbb{R}^d$, and $\phi\in C_c^\infty(\mathbb{R}\times\mathbb{R}^d)$.

3. ...AND NONLOCAL BOUNDARY VALUE PROBLEMS

3.1. **Fractional Laplacian and friends.** Recall $d \in \mathbb{N} := \{1, 2, \ldots\}$, $\alpha \in (0, 2)$, and

$$\nu(x) := c_{d,\alpha} |x|^{-d-\alpha}, \quad x \in \mathbb{R}^d.$$

The constant $c_{d,\alpha}$ is such that

$$|\xi|^\alpha = \int_{\mathbb{R}^d} (1 - \cos \xi \cdot x) \nu(x) \mathrm{d}x\,, \quad \xi \in \mathbb{R}^d.$$

Recall $\nu(x, y) := \nu(y - x) = c_{d,\alpha} |y - x|^{-d-\alpha}$. We interpret $\nu(x, y)dy$ as intensity of jumps of the *isotropic α-stable Lévy proces* on $\mathbb{R}^d$, which we will now denote $(X_t, t \geq 0)$. For $u \in C_c^2(\mathbb{R}^d)$,

$$\begin{aligned}\Delta^{\alpha/2} u(x) &= \lim_{\epsilon \to 0^+} \int_{\{|y-x| > \epsilon\}} \big[u(y) - u(x)\big] \nu(x, y) \mathrm{d}y \\ &= \tfrac{1}{2} \int_{\mathbb{R}^d} \big[u(x + z) + u(x - z) - 2u(x)\big] \nu(z)\, \mathrm{d}z, \quad x \in \mathbb{R}^d.\end{aligned}$$

3.2. **Transition semigroup (back to time-homogeneous notation).** Recall that, by the Lévy–Khinchine formula, there are smooth probability densities with $p_t * p_s = p_{t+s}$ and

$$\int_{\mathbb{R}^d} \mathrm{e}^{i\xi \cdot x} p_t(x) \mathrm{d}x = \mathrm{e}^{-t|\xi|^\alpha}, \quad \xi \in \mathbb{R}^d.$$

We denote $p_t(x, y) := p_t(y - x)$, for $t > 0$, $x, y \in \mathbb{R}^d$. Then,

$$p_t(x, y) = t^{-d/\alpha} p_1(t^{-1/\alpha}(x - y)) \approx t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}}.$$

We get a Feller semigroup of operators (on $C_0(\mathbb{R}^d)$), see [47] or [26], denoted

$$P_t f(x) := \int_{\mathbb{R}^d} f(y) p_t(x, y) \mathrm{d}y, \quad x \in \mathbb{R}^d,\ t \geq 0,$$

with $\Delta^{\alpha/2}$ as generator. Of course, $P_t P_s = P_{t+s}$, $s, t > 0$.

Consider the space $\mathcal{D}([0,\infty))$ of cádlág functions $\omega : [0,\infty) \to \mathbb{R}^d$. On $\mathcal{D}([0,\infty))$, we denote $X_t(\omega) := \omega_t$, $t \geq 0$; $X_{t-} := \lim_{s\uparrow t} X_s$. We also define measures $\mathbb{P}^x$, $x \in \mathbb{R}^d$, as follows:
For $x \in \mathbb{R}^d$, $0 < t_1 < t_2 < \ldots < t_n$ and $A_1, A_2, \ldots, A_n \subset \mathbb{R}^d$,

$$\begin{aligned}
&\mathbb{P}^x(X_{t_1} \in A_1, \ldots, X_{t_n} \in A_n) = \mathbb{P}^x(\omega_{t_1} \in A_1, \ldots, \omega_{t_n} \in A_n) \\
&:= \int_{A_1} \mathrm{d}x_1 \int_{A_2} \mathrm{d}x_2 \ldots \int_{A_n} \mathrm{d}x_n \, p_{t_1}(x, x_1) p_{t_2-t_1}(x_1, x_2) \cdots p_{t_n-t_{n-1}}(x_{n-1}, x_n).
\end{aligned}$$

We let $\mathbb{E}^x$ be the corresponding integration. We call $(X_t, \mathbb{P}^x)$ the isotropic α-stable Lévy process in $\mathbb{R}^d$. It is strong Markov.

3.3. **The first exit time.** We fix D, a nonempty open bounded Lipschitz subset of $\mathbb{R}^d$. The *time of the first exit* of X from D is

$$\tau_D := \{t > 0 : X_t \notin D\}.$$

We will consider the random variables τ_D, X_{τ_D-} and X_{τ_D}. We have $\mathbb{P}^x(\tau_D = 0) = 1$ for $x \in \partial D$. Also, $\mathbb{P}^x(X_{\tau_D} \in \partial D) = 0$ for $x \in D$.

We want to *reflect* X_t at $t = \tau_D$ *back* to D.

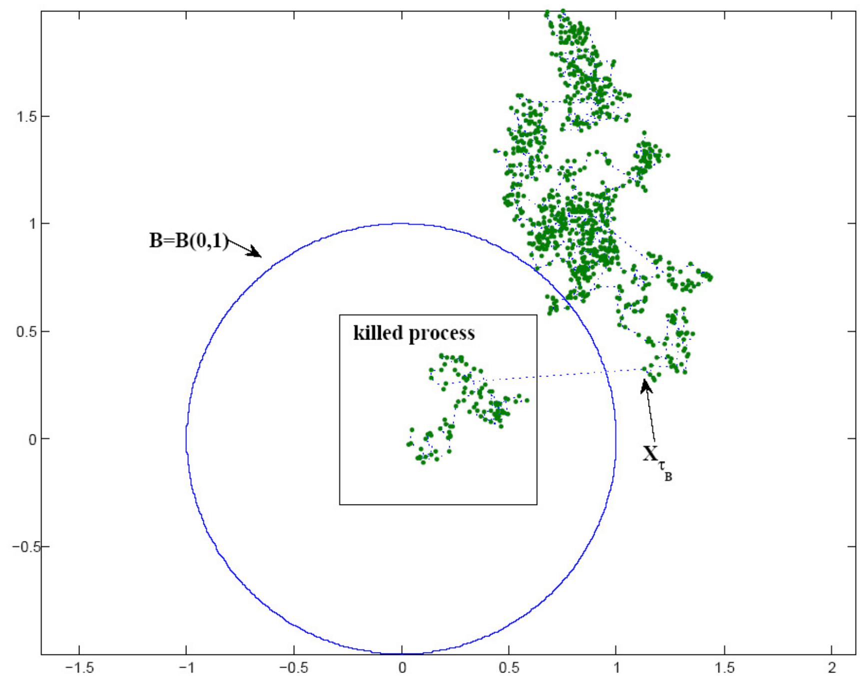

FIGURE 1. Trajectory of the izotropic α-stable Lévy process leaving the unit disc on the plane, $\alpha = 1.8$.

3.4. **Killed semigroup and Ikeda-Watanabe formula.** For $t > 0$, $x \in D$, and suitable functions f, we let

$$P_t^D f(x) := \mathbb{E}^x\big[t < \tau_D;\ f(X_t)\big] =: \int_D f(y) p_t^D(x,y)\mathrm{d}y.$$

This *killed semigroup* (P_t^D) is (strong) Feller: $P_t^D B_b(D) \subset C_0(D)$.

The I-W formula describes the law of $(\tau_D, X_{\tau_D -}, X_{\tau_D})$, for $x \in D$:

$$\mathbb{P}^x[\tau_D \in J, X_{\tau_D -} \in A, X_{\tau_D} \in B] = \int_J\int_B\int_A p_u^D(x,y)\nu(y,z)\mathrm{d}y\mathrm{d}z\mathrm{d}u.$$

Here $J \subset [0,\infty)$, $A \subset D$, $B \subset D^c$. We may interpret $p_u^D(x,y)$ as occupation time density.

3.5. **The (tentative) reflections.** We want a Markov process $(Y_t, t \geq 0)$ equal to X until τ_D, but at τ_D we will perform a *reflection*: instead of

$z = X_{\tau_D} \in D^c$, we let $Y_{\tau_D} = y \in D$ with distribution $\mu(z, \mathrm{d}y)$. This yields jump intensity

$$\gamma(x, \mathrm{d}y) := \nu(x, \mathrm{d}y) + \int_{D^c} \nu(x, dz)\mu(z, \mathrm{d}y) \quad \text{on } D. \tag{3.1}$$

(1) Is there such a thing?

(2) How to construct the corresponding semigroup $(K_t, t > 0)$ and describe its long-time behavior?

(3) What about the *generator* and *boundary conditions*?

3.6. **Tightness assumption.** The outcome of [20] is (just) a conservative exponentially asymptotically stable Markovian semigroup $(K_t, t \ge 0)$, with γ as the integro-differential kernel of generator. For this we make the following assumptions on D and μ:

D is open nonempty bounded Lipschitz set in $\mathbb{R}^d$. Let $\mu : D^c \times \mathscr{B}(D) \to [0, 1]$ be such that $\mu(z, \cdot)$, $z \in D^c$, are weakly continuous tight Borel probability measures on D: for each $\epsilon > 0$ there exists $H \Subset D$ with $\mu(z, H) \ge 1 - \epsilon$ for $z \in D^c$.

We will use the notation

$$\nu \mathbf{1}_{D^c}\mu(v, W) := \int_{D^c} \nu(v, z)\mu(z, W)\mathrm{d}z\,, \quad v \in D, W \subset D.$$

3.7. **Some background on "reflecting".** Similar "reflections" appeared first in Feller [30] for one-dimensional diffusions, called *instantaneous return processes* with non-local boundary conditions. Ikeda, Nagasawa, Watanabe [40], Sharpe [48], Werner [52] deal with "piecing together", "resurrection", "concatenation".

Further (multidimensional) developments: Ben-Ari and Pinski [6], Arendt, Kunkel, and Kunze [2], Taira [49].

For jump processes, one can make Y_{τ_D} depend on X_{τ_D-} and X_{τ_D}:

E.g., KB, Burdzy and Chen [8] propose the censored processes, with the reflection back to X_{τ_D-}. Barles, Chasseigne, Georgelin and Jakobsen [4] discuss geometric reflections depending on $(X_{\tau_D-}, X_{\tau_D})$ for the half-space.

Dipierro, Ros-Oton and Valdinoci [28] essentially postulate $\mu(z, \mathrm{d}y) = \nu(z, \mathrm{d}y)/\nu(z, D)$. However, they discuss Neumann-type problems, not the semigroup or Markov process. See also Felsinger, Kassmann and Voigt [31]. Vondraček [51] proposes a variant of [28, 31].

Palmowski, Grzywny, Szczypkowski study "resetting" (forthcoming).

KB, Fafuła, Sztonyk deal with the Servadei-Valdinoci model (forthcoming).

Bobrowski [7] describes (a limiting case of) "concatenation" in "geometric graphs".

3.8. **Objects related to X.** The *Green function*:

$$G_D(x,y) := \int_0^\infty p_t^D(x,y)\,\mathrm{d}t, \quad x,y \in D.$$

The *expected exit time*:

$$\mathbb{E}^x \tau_D = \int_D G_D(x,y)\,\mathrm{d}y, \quad x \in D.$$

The *survival probability*:

$$\begin{aligned}\mathbb{P}^x(\tau_D > t) &= \int_t^\infty \mathrm{d}s \int_D \mathrm{d}v \int_{D^c} \mathrm{d}z\, p_s^D(x,v)\nu(v,z)\\ &= \int_D p_t^D(x,y)\,\mathrm{d}y, \quad t>0,\, x\in D.\end{aligned}$$

In particular, for all $t > 0$, $x \in D$,

$$\int_D p_t^D(x,y)\mathrm{d}y + \int_0^t \mathrm{d}s \int_D \mathrm{d}v \int_{D^c} \mathrm{d}z\, p_s^D(x,v)\nu(v,z) = 1. \tag{3.2}$$

3.9. **Construction of the semigroup** $(K_t, t > 0)$**.** This follows [15] and [23], as discussed above: For $t > 0$, $x, y \in D$, $n \in \mathbb{N}$, we let $k_t(x,y) := \sum_{n=0}^\infty p_n(t,x,y)$, where

$$\begin{aligned}p_0(t,x,y) &:= p_t^D(x,y),\\ p_n(t,x,y) &:= \int_0^t \mathrm{d}s \int_D \mathrm{d}v \int_D p_{n-1}(s,x,v)\nu \mathbf{1}_{D^c}\mu(v,\mathrm{d}w)p_0(t-s,w,y).\end{aligned}$$

In our notation of nonlocal Schrödinger perturbations (of kernels operating on space-time),

$$K = \sum_{n=0}^\infty (P^D\, \nu \mathbf{1}_{D^c}\mu)^n P^D.$$

Corollary 3.1. *$\int_D k_t(x,y)k_s(y,z)\mathrm{d}y = k_{t+s}(x,z)$ for all $t > 0$, $x, y \in D$.*

For $f \in B_b(D)$, we let $K_t f(x) := \int_D f(y)k_t(x,y)dy$, where $t > 0$, $x \in D$.

3.10. **Main results.**

Theorem 3.2. *$\int_D k_t(x,y)\mathrm{d}y = 1$ for all $t > 0$, $x \in D$.*

Hints: The easy part: $K_t\mathbf{1}(x) = k_t(x, D) := \int_D k_t(x, y)\mathrm{d}y \le 1$.
Indeed, $p_0(t, x, D) := \int_D p_t^D(x, y)dy \le 1$. Then,

$$\begin{aligned} p_1(t,x,D) &:= \int_0^t \mathrm{d}s \int_D \mathrm{d}v \int_D p_s^D(x,v)\nu\mathbf{1}_{D^c}\mu(v,\mathrm{d}w)p_{t-s}^D(w,D) \\ &\le \int_0^t \mathrm{d}s \int_D \mathrm{d}v p_s^D(x,v)\nu(v,D^c), \end{aligned}$$

so, by (3.2), $p_0(t, x, D) + p_1(t, x, D) \le 1$. Similarly, for all $n \in \mathbb{N}$,

$$\sum_{k=0}^{n} p_n(t,x,D) \le 1.$$

For deeper results we use there lower bounds for *fixed* $t > 0$:

$$\begin{aligned} &p_0(t,x,D) + p_1(t,x,D) \ge c > 0, \quad x \in D, \\ &k_t(x,y) \ge \delta > 0, \quad x \in D,\ y \in H \ \text{ for some } H \Subset D \text{ with } |H| > 0. \end{aligned}$$

They follow from known bounds of p^D. (We will discuss some of these in the last section.)
The second bound is a Dobrushin-type condition, which yields exponential egodicity, as follows.

Theorem 3.3. *There is a unique stationary distribution κ for (K_t). Moreover, there exist $M, \omega \in (0,\infty)$ such that for every probability measure ρ on D,*

$$\|\rho K_t - \kappa\|_{TV} \le M e^{-\omega t}, \quad t > 0.$$

3.11. **Generator and boundary conditions.** Given a function $f \in C_b(D)$, we let

$$f_\mu(x) := \begin{cases} f(x), & \text{for } x \in D, \\ \mu(x,f), & \text{for } x \in D^c, \end{cases}$$

where

$$(\mu f)(z) := \mu(z,f) := \int_D \mu(z,\mathrm{d}y)f(y), \quad z \in D^c.$$

We define the space $C_\mu(D)$ by

$$C_\mu(D) := \{f \in C_b(D) : f_\mu \in C_b(\mathbb{R}^d)\}.$$

Proposition 3.4. *$K_t f \to f$ uniformly as $t \to 0$ if, and only if, $f \in C_\mu(D)$.*

We consider the Laplace transform (resolvent) R_λ of K_t, defined by

$$R_\lambda := \int_0^\infty e^{-\lambda t} K_t \mathrm{d}t, \quad \lambda > 0,$$

and relate it to the Laplace transform R_λ^D of P^D. By perturbation formula,

$$K_t = P^D + \int_0^t P_s \nu \mathbf{1}_{D^c} \mu K_{t-s} \mathrm{d}s = P^D + \int_0^t K_s \nu \mathbf{1}_{D^c} \mu P_{t-s}^D \mathrm{d}s,$$

which leads to

$$R_\lambda = R_\lambda^D + R_\lambda^D \nu \mathbf{1}_{D^c} \mu R_\lambda = R_\lambda^D + R_\lambda \nu \mathbf{1}_{D^c} \mu R_\lambda^D.$$

The generator A of K_t is defined on $D(A) := R_\lambda(C_b(D))$ by $A := \lambda - R_\lambda^{-1}$.

Theorem 3.5. *For $u, f \in C_b(D)$, the following are equivalent:*

(1) *$u \in D(A)$ and $Au = f$.*

(2) *$u \in C_\mu(D)$ and, with $\gamma := \nu + \nu \mathbf{1}_{D^c} \mu$ as kernels on D, given by (3.1),*

$$f(x) = \lim_{\epsilon \to 0^+} \int_{\{|y-x|>\epsilon\} \cap D} (u(y) - u(x)) \gamma(x, \mathrm{d}y), \qquad x \in D.$$

3.12. **Issues.**

(1) (K_t) is a C_b-semigroup and has the strong Feller property, but it is not Feller (on $C_0(D)$) nor symmetric nor bounded on $L^2(D)$ in general.

(2) The existence of (Y_t) requires a separate approach. (Not yet done, but concatenation of right processes applies.) Also called piecing-out, resetting, resurrection, instantaneous return, Neumann-type conditions.

(3) Test functions $C_c^\infty(D)$ are not in the domain of the generator.

(4) The range of the resolvent is a specific function space with boundary condition expressed via μ.

(5) It is convenient to use the Dynkin operator as generator.

(6) This is about constructing new semigroups by positive nonlocal perturbations of P_t^D. The perturbing kernel "defines" boundary conditions.

(7) Reflected trajectories in models without tightness can accumulate at the boundary.

3.13. **Summary.** We propose in [20] a framework for constructing semigroups with specific reflection mechanism from the killed semigroup. The restriction to $\Delta^{\alpha/2}$ can be easily relaxed, but the tightness condition is more tricky.

This area of research is motivated by the Neumann-type boundary-value problems [4, 28] and by the problem of piecing-out or concatenation of Markov processes in the sense of Ikeda, Nagasawa and Watanabe [40], Sharpe [48] and Werner [52].

Besides construction, questions arise on large-time and boundary behavior of the semigroup (process) and on applications to nonlocal differential equations with those boundary conditions.

4. Limits in the homogeneous setting

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Department of Mathematics, Wrocław University of Science Technology, Poland
Email address: `krzysztof.bogdan@pwr.edu.pl,`

Department of Mathematics, Wrocław University of Technology, Poland
Email address: `karol.szczypkowski@pwr.edu.pl`