1A. Markov processes, heat kernels, Green functions and harmonic measures: hitchhiker's guide to definitions, results and connections *(Fractional Laplacian)*

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Plan

1. Operators with maximum principle
2. Fractional Laplacian, semigroup and process
3. Dirichlet heat kernel

Every operator A on $C_c^\infty(\mathbb{R})$ with positive maximum principle:

$$\sup_{y\in\mathbb{R}} \varphi(y) = \varphi(x) \geqslant 0 \quad \textit{implies} \quad A\varphi(x) \leqslant 0\,,$$

is given, by Courrége, uniquely in the form

$$\begin{aligned} A\varphi(x) \;&=\; \sum_{i,j=1}^{d} a_{ij}(x) D_{x_i} D_{x_j} \varphi(x) + b(x)\nabla\varphi(x) - c(x)\varphi(x) \\ &\quad + \int_{\mathbb{R}} \left(\varphi(x+y) - \varphi(x) - y\nabla\varphi(x)\, 1_{|y|<1}\right) \nu(x,dy)\,. \end{aligned}$$

Here for every x, $a(x) := (a_{ij}(x))_{i,j=1}^n$ is a nonnegative definite real symmetric matrix, $b(x) := (b_i(x))_{i=1}^d$ has real coordinates, $c(x) \geqslant 0$, and $\nu(x,\cdot)$ is a Lévy measure: $\int_{\mathbb{R}^d} \min(|y|^2,\, 1)\, \nu(x,dy) < \infty\,$.

Assume $a = 0$, $b = 0$, $c = 0$, $\nu(x, dy) = \nu(dy) = \nu(-dy)$. We construct a corresponding semigroup as follows. For $\varepsilon > 0$, $\nu_\varepsilon := 1_{B(0,\varepsilon)^c}\nu$. Let

$$\begin{aligned} P_t^\varepsilon &= \exp(t(\nu_\varepsilon - |\nu_\varepsilon|\delta_0)) = \sum_{n=0}^{\infty} \frac{t^n \left(\nu_\varepsilon - |\nu_\varepsilon|\delta_0\right)^n}{n!} \\ &= e^{-t|\nu_\varepsilon|} \sum_{n=0}^{\infty} \frac{t^n \nu_\varepsilon^n}{n!}, \quad t > 0\,. \end{aligned}$$

Here $\nu_\varepsilon^n = (\nu_\varepsilon)^{*n}$. We get $P_t^\varepsilon * P_s^\varepsilon = P_{s+t}^\varepsilon\,, \quad s,\, t > 0\,$. The Fourier transform of P_t^ε is

$$\int e^{iuy} P_t^\varepsilon(dy) = \exp\left(t \int (e^{iuy} - 1)\nu_\varepsilon(dy)\right), \quad u \in \mathbb{R}\,.$$

The measures P_t^ε weakly converge to a probability measure P_t as $\varepsilon \to 0$. We call ν the *Lévy measure* of the semigroup $\{P_t\,,\ t \geqslant 0\}$.

We fix $d \geqslant 1$ and $0 < \alpha < 2$. Let

$$\nu(y) := c|y|^{-d-\alpha}, \quad y \in \mathbb{R}^d .$$

Here c is such that

$$\int_{\mathbb{R}^d} [1 - \cos(\xi \cdot y)]\, \nu(y) dy = |\xi|^\alpha , \quad \xi \in \mathbb{R}^d .$$

Define $\quad p_t(x) := (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} e^{-t|\xi|^\alpha} d\xi , \quad x \in \mathbb{R}^d , \ t > 0 .$

Scaling: $p_t(x) = t^{-d/\alpha} p_1(t^{-1/\alpha} x)$.

Let $P_t f := f * p_t$. We have, e.g., for $f \in C_0^2(\mathbb{R}^d)$,

$$\lim_{t\to 0^+} \frac{P_t f(x) - f(x)}{t} = \lim_{\varepsilon \to 0+} \int\limits_{|y|>\varepsilon} [f(x+y) - f(x)]\nu(dy) =: \Delta^{\alpha/2} f(x).$$

Let Ω be the class of càdlàg functions $X : [0, \infty) \to \mathbb{R}^d$.

For $x \in \mathbb{R}^d$ we define probability $\mathbb{P}_x$ on Ω by

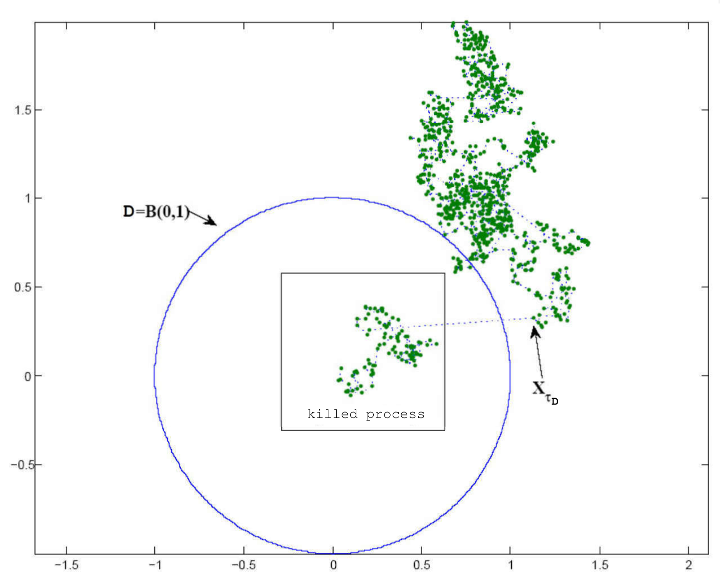
$$\mathbb{P}_x(X_{t_1} \in B_1, \ldots, X_{t_n} \in B_n) := \int_{B_1} p_{t_1}(x_1 - x) \cdots \int_{B_n} p_{t_n - t_{n-1}}(x_n - x_{n-1})\, dx_n \cdots dx_1.$$

Expectation: $\mathbb{E}_x := \int_\Omega d\mathbb{P}_x$.

Let $D \subset \mathbb{R}^d$ be open; define

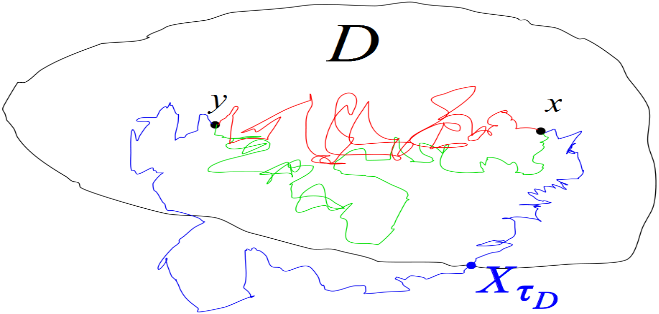
$$\tau_D := \inf\{t > 0 : X_t \notin D\} \qquad \text{(first exit/ruin time).}$$

Simulated trajectory $t \mapsto X_t$, for $\alpha = 1.8$, $d = 2$



Dirichlet heat kernel=transition density of the process killed off D

$$p_t^D(x,y) := p_t(y-x) - \mathbb{E}_x\left[\tau_D \leqslant t;\ p_{t-\tau_D}(y - X_{\tau_D})\right]$$



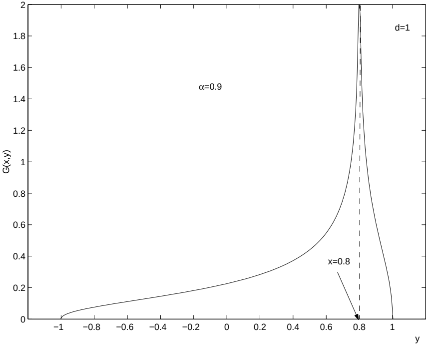
Then, $P_t^D f(x) := \mathbb{E}_x[t < \tau_D;\ f(X_t)] = \int_{\mathbb{R}^d} f(y) p_t^D(x,y)dy$,

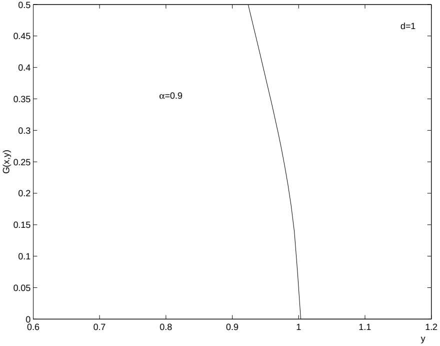
$G_D(x,y) := \int_0^\infty p_t^D(x,y)dt.$

M. Riesz' formula for the ball

M. Riesz 1938; Blumenthal, Getoor, Ray 1961:
Let $x, y \in B(0,1) \subset \mathbb{R}^d$, and $w := (1-|x|^2)(1-|y|^2)/|x-y|^2$. Then,

$$G_{B(0,1)}(x,y) = \mathcal{B}_{d,\alpha}\,|x-y|^{\alpha-d} \int_0^w \frac{r^{\alpha/2-1}}{(r+1)^{d/2}}\,dr\,.$$





Connection to generator:

For $s \in \mathbb{R}$, $x \in \mathbb{R}^d$, $\phi \in C_c^\infty(\mathbb{R} \times D)$, we have

$$\int_s^\infty \int_D p_{t-s}^D(x,z) \left(\partial_t + \Delta_y^{\alpha/2}\right) \phi(t,y)\, dy\, dt = -\phi(s,x)\,.$$

Considering $\phi(t,y) = \varphi(y)$, we get

$$\int_{\mathbb{R}^d} G_D(x,y) \Delta^{\alpha/2} \varphi(y) dy = -\varphi(x)\,.$$

Glossary of formulas for Dirichlet conditions (killing/stopping on D^c):

$G_D(x,y) := \int_0^\infty p_t^D(x,y)dt$,

$G_D f(x) := \int_{\mathbb{R}^d} G_D(x,y)f(y)dy = \mathbb{E}_x \int_0^{\tau_D} f(X_t)dt$,

$\omega_D^x(A) := \mathbb{P}_x(X_{\tau_D} \in A)$, etc.

$u(x) := \mathbb{E}_x g(X_{\tau_D})$ is harmonic/a solution to Dirichlet problem:

$$\Delta^{\alpha/2} u = 0 \text{ on } D, \quad u = g \text{ on } D^c.$$

$u(x) := \mathbb{E}_x g(X_{\tau_D}) - G_D f(x)$ solves inhomogeneous Dirichlet problem:

$$\Delta^{\alpha/2} u = f \text{ on } D, \quad u = g \text{ on } D^c.$$

Glossary of formulas for Dirichlet conditions (continued).

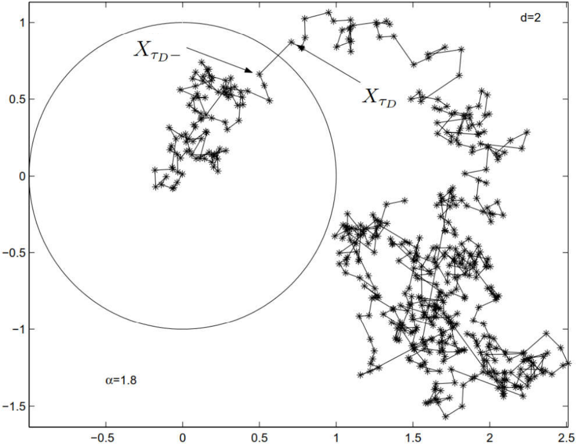
If $u(x) = \mathbb{E}_x g(X_{\tau_D}) - G_D\varphi(\cdot, u(\cdot))(x)$, $x \in D$, then it solves

$$\Delta^{\alpha/2} u(x) = \varphi(x, u(x)), \quad u = g \text{ on } D^c.$$

Further, $u(t,x) := \int_{\mathbb{R}^d} p_t^D(x,y) f(y) dy = \mathbb{E}_x \left[t < \tau_D;\ f(X_t) \right]$ solves

$$\partial_t u = \Delta_x^{\alpha/2} u \text{ on } (0,\infty) \times D,\ u(0+,\cdot) = f \text{ on } D,\ u(t,\cdot) = 0 \text{ on } D^c.$$

Etc.



Ikeda-Watanabe formula (for Lipschitz D):

$$\mathbb{P}_x(X_{\tau_D-} \in dy, X_{\tau_D} \in dz, \tau_D \in dt) = p_t^D(x,y)\,\nu(z-y)dy\,dz\,dt,$$

$$\omega_D^x(dz) = \int_D G_D(x,y)\nu(z-y)dy\,dz.$$

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