E N T R O P Y   S T R U C T U R E
ENTROPY VERSUS RESOLUTION

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Based on the papers [B-D] and [D]

MOTIVATION

Entropy measures exponential complexity in a topological dynamical system:

- topological entropy – very crude,

- entropy function on invariant measures
– tells "where" the complexity is located.

But, neither tells
- how and where the complexity emerges on refining scales.

There are many different ways of computing entropy of invariant measures in a general topological dynamical system. Each of the methods involves computation of a sequence of functions (on measures) reflecting the complexity "detectable" in a certain finite "resolution" (scale). The entropy function is then obtained as the limit as the resolution refines. Below we quote various definitions, all leading to the same function $h(\mu)$:

Entropy with respect to finite partitions

$$h(\mu,\mathcal{A}) = \lim_{n\to\infty} \tfrac{1}{n} H_\mu(\mathcal{A}^n)$$
$$h(\mu) = \lim_{k\to\infty} h(\mu,\mathcal{A}_k).$$

Katok's entropy
(1980)

Let $\mathcal{B}^{n,\epsilon}_{\sigma,\mu}$ be any collection of (n,ϵ)-balls whose union has μ-measure larger than σ.

$$h(\mu,\epsilon|\sigma) := \overline{\lim_{n\to\infty}}\ \tfrac{1}{n}\log\min\#\mathcal{B}^{n,\epsilon}_{\sigma,\mu}$$
$$h(\mu) = \lim_{\epsilon\to 0} h(\mu,\epsilon|\sigma) \quad \text{(for any } 0<\sigma<1).$$

Brin-Katok local entropy
(1983)

$$I(n,\mu,x,\epsilon) := -\log\mu(\mathbf{B}^{(n,\epsilon)}_x)$$
$$h(\mu,x,\epsilon) := \overline{\lim_{n\to\infty}}\ \tfrac{1}{n} I(n,\mu,x,\epsilon)$$
$$h(\mu,\epsilon) = h(\mu,x,\epsilon) \quad \mu\text{-almost everywhere}$$
$$h(\mu) = \lim_{\epsilon\to 0} h(\mu,\epsilon).$$

Romagnoli's entropy
(2001)

For an open cover $\mathcal{U}$ of X let
$h(\mu,\mathcal{U}) := \lim_{n\to\infty} \frac{1}{n} \inf\{H_\mu(\mathcal{A}) : \mathcal{A} \succcurlyeq \mathcal{U}^n\}$.
$h(\mu) = \lim_{k\to\infty} h(\mu,\mathcal{U}_k)$.

Ornstein-Weiss type entropy
(1993, modified by T.D. 2002)

$R(n,\epsilon,x) := \min\{k > 0 : T^k(x) \in \mathbf{B}_x^{(n,\epsilon)}\}$
$h(x,\epsilon) := \overline{\lim_{n\to\infty}} \frac{1}{n} \log R(n,\epsilon,x)$
$h(\mu,\epsilon) = h(x,\epsilon)$ μ-almost everywhere
$h(\mu) = \lim_{\epsilon\to 0} h(\mu,\epsilon)$.

Modified Bowen's entropy
(T.D.'s notion 2002)

For a measurable set $F \subset X$ let
$H(n,\epsilon|F) := \log\max\{\#E :\ E$ is a (d^n,ϵ)-separated set within $F\}$
$h(\epsilon|F) := \overline{\lim_{n\to\infty}} \frac{1}{n} H(n,\epsilon|F)$
$h(\mu,\epsilon) := \inf\{h(\epsilon|F) : \mu(F) > \sigma\}$ (does not depend on $0 < \sigma < 1$)
$h(\mu) = \lim_{\epsilon\to 0} h(\mu,\epsilon)$.

Newhouse's local entropy
(1990)

Let $F \subset X$ be a measurable set. Define
$H(n,\delta|x,F,\epsilon) := \log\max\{\#E : E$ is a (d^n,δ)-separated set within $F \cap \mathbf{B}_x^{(n,\epsilon)}\}$
$H(n,\delta|F,\epsilon) := \sup_{x\in F} H(n,\delta|x,F,\epsilon)$
$h(\delta|F,\epsilon) := \overline{\lim_{n\to\infty}} \frac{1}{n} H(n,\delta|F,\epsilon)$
$h(X|F,\epsilon) := \lim_{\delta\to 0} h(\delta|F,\epsilon)$
$h(X|\mu,\epsilon) := \lim_{\sigma\to 1} \inf\{h(X|F,\epsilon) : \mu(F) > \sigma\}$
$h(\mu) = \lim_{\epsilon\to 0} (h(\mu) - h(X|\mu,\epsilon))$.

Modified Misiurewicz's conditional entropy

(1976, modified by T.D. 2002)

$$\mathbf{H}(n,\mathcal{U}|x,\mathcal{V}) := \log\max\{\#\text{minimal subfamily of } \mathcal{U}^n \text{ covering } V_x^n\}$$
$$\mathbf{h}(\mathcal{U}|x,\mathcal{V}) := \lim_{n\to\infty} \tfrac{1}{n}\mathbf{H}(n,\mathcal{U}|x,\mathcal{V})$$
$$\mathbf{h}(X|x,\mathcal{V}) := \sup_{\mathcal{U}} \mathbf{h}(U|x,\mathcal{V})$$
$$\mathbf{h}(X|\mathcal{V}) := \sup_{x\in X} \mathbf{h}(X|x,\mathcal{V})$$
$$\mathbf{h}^* := \lim_{k\to\infty} \mathbf{h}(X|\mathcal{V}_k)$$
$$\mathbf{h}(\mathcal{U}|\mu,\mathcal{V}) = \mathbf{h}(\mathcal{U}|x,\mathcal{V}) \quad \mu\text{-almost everywhere}$$
$$\mathbf{h}(X|\mu,\mathcal{V}) := \sup_{\mathcal{U}} \mathbf{h}(\mathcal{U}|\mu,\mathcal{V})$$
$$h(\mu) \neq \lim_{k\to\infty} (h(\mu) - \mathbf{h}(X|\mu,\mathcal{V}_k)).$$

What do these sequences have in common? Is there a unified approach which includes all of them? Is there a deeper sense behind the emergence of entropy in ever refining resolution?

- The Misiurewicz's topological tail entropy $\mathbf{h}^*$ is roughly the limit (as resolution refines) of the following: how much entropy (globally) remains "undetected" at a given scale. This is a rather crude measurement, but it is part of the phenomenon which we want to capture.
- In a symbolic extension (equivalently, in an expansive extension) every complexity, even the microscopic one, has to be "magnified" so it becomes detectable at the coarse resolution determined by the expansive constant, often leading to increased entropy. Thus the entropy theory of symbolic extensions is also related to the phenomenon under study.
- In zero-dimensional systems we have used a refining sequence of clopen partitions $\mathcal{A}_k$ ($k \in \mathbb{N}$) and the sequence of functions

$$h_k : \mathcal{M}_T \to [0,\infty),$$

where

$$h_k(\mu) = h_\mu(T,\mathcal{A}_k).$$

The interesting phenomena depended on the "faults of uniformity" of the convergence $h_k \to h$. It was important, that the functions h_k and $h_{k+1} - h_k$ were affine and upper-semicontinuous.

- In general spaces none of the mentioned definitions of entropy leads to a sequence with all these properties.
- Even if we could define such (h_k), it would not be a topological invariant.

SOLUTION

We introduce a very simple equivalence relation among nondecreasing sequences of real-valued functions (abstractly, on any domain), and we define the *entropy structure* of a topological dynamical system (X,T) as a carefully specified equivalence class of sequences of functions on $\mathcal{M}_T$.

The entropy structure so defined satisfies the following:

- it is a topological invariant,
- it covers most of known entropy invariants, including $\mathbf{h}^*$ and the symbolic extension entropy functions,
- it includes most of the sequences arising from the mentioned earlier methods of computing entropy.

DETAILS

Uniform equivalence

Definition (T.D.)**.** Let $\mathcal{F} = (f_k)$ and $\mathcal{F}' = (f'_k)$ be two non-decreasing sequences of functions on an arbitrary domain $\mathcal{P}$. We say that $\mathcal{F}'$ *uniformly dominates* $\mathcal{F}$ (we write $\mathcal{F}' \overset{uni}{\geq} \mathcal{F}$) if

$$\forall_k \ \forall_\epsilon \ \exists_{k'} \quad f'_{k'} > f_k - \epsilon.$$

We say that $\mathcal{F}$ and $\mathcal{F}'$ are *uniformly equivalent* if both

$$\mathcal{F}' \overset{uni}{\geq} \mathcal{F} \ \text{ and } \ \mathcal{F} \overset{uni}{\geq} \mathcal{F}'.$$

Some notation

We will consider nonnegative functions defined on a compact domain $\mathcal{P}$. For a bounded function f we let

$$\widetilde{f} := \inf\{g : g \geq f, g \text{ continuous}\} \ \ \text{(the } \textit{u.s.c. envelope}\text{)},$$
$$\dddot{f} := \widetilde{f} - f \ \ \text{(the } \textit{defect}\text{)}.$$

If f is unbounded then $\widetilde{f} \equiv \dddot{f} \equiv \infty$.

Superenvelopes

Definition (T.D.)**.** Let $\mathcal{F} = (f_k)_{k\in\mathbb{N}}$ be a nondecreasing sequence of functions defined on a compact space $\mathcal{P}$, with a bounded limit f. By a *superenvelope* of $\mathcal{F}$ we mean any function $E \geq f$ defined on $\mathcal{P}$, which, at every $x \in \mathcal{P}$, satisfies the condition:

$$\lim_{k\to 0} \dddot{(E - f_k)}(x) = 0.$$

In any case (including f unbounded or infinite), we admit the constant ∞ function as a superenvelope of $\mathcal{F}$.

Lemma. *If f is bounded then the function $E - f$ is u.s.c.*

Definition.
Denote by $\mathrm{E}\mathcal{F}$ the infimum of all superenvelopes of $\mathcal{F}$. This function is either bounded or it is the constant ∞.

Lemma. E$\mathcal{F}$ *is itself a superenvelope of* $\mathcal{F}$.

Lemma. *Let* $\mathcal{F} = (f_k)$ *be such that* $f_{k+1} - f_k$ *is u.s.c. for each* k*, and let* $E \geq f$ *be a function on* $\mathcal{P}$. *Then* E *is a bounded superenvelope of* $\mathcal{F}$ *if and only if* $E - f_k$ *is u.s.c. for every* k.

Lemma. *If* $\mathcal{F}$ *defined on a Choquet simplex* $\mathcal{P}$ *has u.s.c. differences and consists of affine functions, then* E$\mathcal{F}$ *coincides with the pointwise infimum of all affine superenvelopes.*

Transfinite sequence, order of accumulation

Definition (Mike Boyle and T.D.)**.** Let $\mathcal{F}$ be a nondecreasing sequence on a compact domain $\mathcal{P}$, with a bounded limit f. Let $\tau_k = f - f_k$. We define the *transfinite sequence associated to* $\mathcal{F}$ by setting

$$u_0 = u_0^{\mathcal{F}} :\equiv 0, \qquad (0)$$

then, for an ordinal α we let

$$u_{\alpha+1} = u_{\alpha+1}^{\mathcal{F}} := \lim_{k\to\infty} \widetilde{u_\alpha + \tau_k}. \qquad (\alpha+1)$$

Finally, for a limit ordinal β let

$$u_\beta = u_\beta^{\mathcal{F}} := \widetilde{\sup_{\alpha<\beta} u_\alpha}. \qquad (\beta)$$

If f is unbounded or infinite, we set $u_\alpha \equiv \infty$ for all $\alpha \geq 1$.

Definition. The smallest ordinal α_0 for which $u_{\alpha_0+1} = u_{\alpha_0}$ (and then automatically $u_\alpha = u_{\alpha_0}$ for every $\alpha \geq \alpha_0$) will be called the *order of accumulation of* $\mathcal{F}$. This is always a countable ordinal.

Lemma. (Mike Boyle and T.D.) *Let* $\mathcal{F}$ *be an increasing sequence of u.s.c. functions with u.s.c. differences, converging to a bounded limit* f. *Then*

$$\mathrm{E}\mathcal{F} = f + u_{\alpha_0}.$$

It is immediately seen that $u_\alpha \leq \alpha u_1$ for any integer α. Thus, with the assumptions of the above lemma, if the order of accumulation α_0 happens to be finite, then

$$\mathrm{E}\mathcal{F} \leq f + \alpha_0 u_1.$$

Theorem. (T.D.) *Let* $\mathcal{F} = (f_k)$ *and* $\mathcal{F}' = (f'_k)$ *be two uniformly equivalent nondecreasing sequences of functions. Then*

- $\lim \mathcal{F} = \lim \mathcal{F}'$,
- $\mathcal{F} \to f$ *uniformly* $\iff$ $\mathcal{F}' \to f$ *uniformly,*
- $u_\alpha^{\mathcal{F}} = u_\alpha^{\mathcal{F}'}$ *for every ordinal* α,
- $\alpha_0^{\mathcal{F}} = \alpha_0^{\mathcal{F}'}$,
- $\mathcal{F}$ *and* $\mathcal{F}'$ *have the same superenvelopes,*
- $\mathrm{E}\mathcal{F} = \mathrm{E}\mathcal{F}'$.

The entropy structure

Theorem. (Mike Boyle and T.D. based on work of E. Lindenstrauss) *Every finite entropy dynamical system (X,T) admits a zero-dimensional principal extension (X',T').*

Definition. By a *reference entropy structure* for a finite entropy dynamical system (X,T) we shall mean the sequence $\mathcal{H}^{ref} = (h_k^{ref})$ of functions on $\mathcal{M}_{T'}$, where $h_k^{ref}(\mu') = h_{\mu'}(T', \mathcal{A}'_k)$ for a refining sequence of clopen partitions $\mathcal{A}'_k$.

By an <u>entropy structure</u> of (X,T) we shall mean any non-decreasing sequence $\mathcal{H} = (h_k)$ of functions defined on $\mathcal{M}_T$ such that for any choice of a zero-dimensional principal extension (X',T') and any choice of clopen partitions $\mathcal{A}'_k$ in X', the lift of $\mathcal{H}$ to $\mathcal{M}_{T'}$ is uniformly equivalent to the corresponding reference entropy structure $\mathcal{H}^{ref}$.

Theorem. (T.D.) *Let $\epsilon_k \to 0$, and let $\mathcal{U}_k$ be a sequence of open covers of X with* $\mathrm{diam}(\mathcal{U}_k) \le \epsilon_k$. *The following sequences are entropy structures:*

- *the Katok's entropy $h(\mu, \epsilon_k|\sigma)$ for any fixed $0 < \sigma < 1$,*
- *the Brin-Katok entropy $h(\mu, \epsilon_k)$,*
- *the Romagoli's entropy $h(\mu, \mathcal{U}_k)$,*
- *the Ornstein-Weiss type entropy $h(\mu, \epsilon_k)$,*
- *the modified Bowen's entropy $h(\mu, \epsilon_k)$,*
- *the (reversed) Newhouse's local entropy $h(\mu) - h(X|\mu, \epsilon_k)$.*

Alternative definition. *Entropy structure* of (X,T) is the uniform equivalence class on $\mathcal{M}_T$ containing any (all) of the above sequences.

The perfect definition of entropy

Definition (T.D.)**.** For a finite family $\mathcal{F}$ of continuous functions on X with values in $[0,1]$ let

$H(\mu, \mathcal{F}) := H(\mu \times \lambda, \mathcal{A}_{\mathcal{F}})$, where λ denotes the Lebesgue measure on the interval.

$h(\mu, \mathcal{F}) := \lim\limits_{n\to\infty} \frac{1}{n} H(\mu, \mathcal{F}^n)$.

- $H(\mu, \mathcal{F})$ is a continuous function of μ.
- $h(\mu, \mathcal{F})$ is an affine u.s.c. function of the invariant measure.
- If $\mathcal{F} \subset \mathcal{G}$ then $h(\mu, \mathcal{G}) - h(\mu, \mathcal{F})$ is a u.s.c. function.
- We can arrange an increasing (wrt. inclusion) sequence of families $\mathcal{F}_k$ such that the partitions $\mathcal{A}_{\mathcal{F}_k}$ refine in the product $X \times [0,1]$.

Theorem. *The sequence $h(\mu, \mathcal{F}_k)$ belongs to the entropy structure.*

Realization theorem

Theorem. (T.D. and Jacek Serafin) *A uniform equivalence class defined on an arbitrary (abstact) metrizable Choquet simplex is (up to affine homeomorphism) an entropy structure for some topological dynamical system (and then also for some minimal zero-dimensional one) if and only if it contains a nondecreasing sequence of nonnegative affine u.s.c. functions with u.s.c. differences.*

Elementary properties

Theorem.

- *If (X,T) is a factor of (Y,S), then $\mathcal{H}^S \overset{uni}{\geq}$ lifted $\mathcal{H}^T$.*
- *The entropy structure is a topological invariant.*
- *If (X',T') is a principal extension of (X,T) then $\mathcal{H}^{T'}$ and lifted $\mathcal{H}^T$ are uniformly equivalent.*
- *If $\mathcal{H}=(h_k)$ is the entropy structure for (X,T) and $m\in\mathbb{N}$ then (mh_k) is the entropy structure for (X,T^m).*

Master invariant theorems

Theorem. (parts by T.D., parts by Mike Boyle and T.D.)

- $h=\lim\mathcal{H}$, $\mathbf{h}_{\text{top}}=\sup h$.
- *The family of all bounded affine superenvelopes E of $\mathcal{H}$ coincides with the family of all extension entropy functions h^π_{ext} of symbolic extensions. In particular,*
- $h_{\text{sex}}=\mathrm{E}\mathcal{H}=h+u_{\alpha_0}$ *and* $\mathbf{h}_{\text{sex}}=\sup\mathrm{E}\mathcal{H}$.
- *The function h_{sex} is attained as h^π_{ext} for a symbolic extension π if and only if $\mathrm{E}\mathcal{H}$ is finite and affine.*
- *The topological tail entropy $\mathbf{h}^*$ equals $\sup u_1$* (!)
- *(X,T) is asymptotically h-expansive $\iff$ $\mathcal{H}$ converges uniformly $\iff$ $\alpha_0=0$ $\iff$ (X,T) has a principal symbolic extension.*

We define the *tail entropy function* by $h^*:=u_1$. At each measure μ it bounds from above the defect of h at μ.

Definition. We define the *order of accumulation of entropy* as the order of accumulation α_0 of the entropy structure $\mathcal{H}$.

Application: if $\alpha_0\in\mathbb{N}$ then the system admits a symbolic extension and

$$h_{sex}\leq h+\alpha_0 h^*,\quad \mathbf{h}_{\text{sex}}\leq\mathbf{h}_{\text{top}}+\alpha_0\mathbf{h}^*.$$

It is worth mentioning that,

- (T.D. and Sheldon Nowhouse) *A typical non-Anosov area-preserving C^1 map on a compact Riemannian manifold has infinite order of accumulation of entropy and* $\mathbf{h}_{\text{sex}}=\infty$ (no symbolic extensions).
- (T.D. and Sheldon Nowhouse) *For $1<r<\infty$, there is a residual subset of an C^r-open set of maps in which the order of accumulation of entropy is infinite and* $\mathbf{h}_{\text{sex}}>\mathbf{h}_{\text{top}}$ (no principal symbolic extensions).
- (Buzzi '97, but also Newhouse '89) *Every C^∞ map is asymptotically h-expansive* (i.e., has order of accumulation zero, i.e., it has a principal symbolic extension).

Wrong sequences

- The sequence $h(\mu, \mathcal{A}_k)$ with $\operatorname{diam}(\mathcal{A}_k) \to 0$ generally DOES NOT belong to the entropy structure.

- The sequence $\mathbf{h}(X|\mu, \mathcal{V}_k)$ (the modified Misiurewicz's conditional entropy) treated as the sequence of tails τ_k leads to the correct function u_1 (which is used to prove that $\mathbf{h}^*$ is indeed a parameter of the entropy structure), but usually it leads to a wrong u_α for $\alpha > 1$, wrong order of accumulation and wrong superenvelopes. This notion also FAILS to comply with the entropy structure (same example).

QUESTIONS

- Both topological entropy $\mathbf{h}_{\text{top}}$ and the topological tail entropy $\mathbf{h}^*$ can be computed without refering to invariant measure. Is the same true for $\mathbf{h}_{\text{sex}}$?
- Can one define a substitute of the entropy structure directly on the space X?
- Is the order of accumulation in C^r systems is at most ω_0 $(1 \le r < \infty)$?
- Is $\mathbf{h}_{\text{sex}}$ finite in C^r systems $(1 < r < \infty)$?
- What other properties can be investigated using the entropy structure or its variations?

References

[B-D] M. Boyle and T. Downarowicz, *The entropy theory of symbolic extensions*, Inventiones Math. **156** (2004), 119–161.

[D] T. Downarowicz, *The entropy structure*, J. d'Analyse (to appear).