Contributions of Hugo Steinhaus to Probability Theory

Wojbor A. WOYCZYŃSKI

Statistics Department and
Center for Stochastic and Chaotic Processes in
Science and Technology

Case Western Reserve University

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Antoni Łomnicki (1881-1941)
Hugo Steinhaus, (1887-1972)

1906/7 Göttingen

"In this time the papers by Georg Cantor, the brillant creator of set theory, were hardly known yet. As the young mathematician of Warsaw Wacław Sierpiński discovered: the position of a point in the plane may be described by only one number, he does not believe it. He wrote a letter to his friend Tadeusz Banachiewicz in Göttingen, This one went to the reading-room, looked out Cantor's corresponding paper and telegraphed to Sierpiński only: 'G. Cantor, Journal für Math, volume,pages...' That awoke our interest. Together with Antoni Łomnicki we went reading eagerly Borel's *Theorie des fonctions.* "

1898 (HS, Znak (1970), trans. H. Girlich)

- Henri Poincaré

Calcul des probabilités , Paris (1912)

“On ne peut guère donner une définition satisfaisante de la *probabilité*”

- Stefan Mazurkiewicz

La théorie des probabilités , in Poradnik dla samouków, Warszawa (1915)

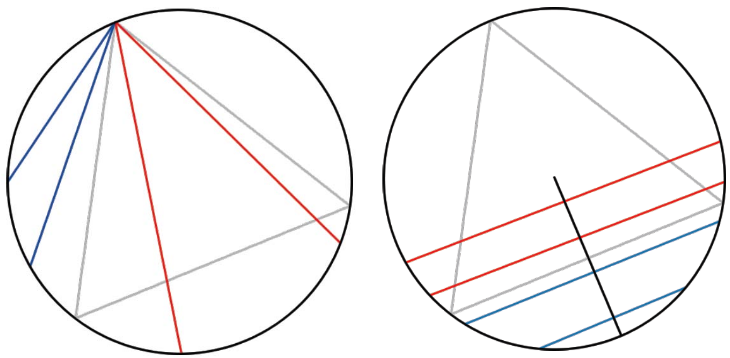
„La théorie des probabilités n'est pas un élément indispensable de l'instruction mathématique; il est toutefois bien désirable pour un mathématicien d'en connaître les principes généraux... Les notions fondamentales y sont incomplètement fixées: elles contiennent beaucoup de difficultés inexpliquées“.

- Richard von Mises

Grundlagen der Wahrscheinlichkeitsrechnung , Math. Z. , Berlin (1919)

"... die Wahrscheinlichkeitsrechnung heute eine mathematische Disziplin nicht ist"

Bertrand paradox :



ŁOMNICKI

- Nuoveaux fondements de la théorie des probabilités (Définition de la probabilité fondée sur la théorie des ensembles), *Fundamenta Math.* 4 (1923), 34- 71, (Signed: Léopol, le 19 Novembre 1920)

STEINHAUS

- Les probabilités dénombrables et leur rapport à la théorie de la mesure, *Fundamenta Math.* 4 (1923), 286-310, (Signed: Goetingue, 18 juin 1922)

They do not cite each other !!!

The same volume of FM contains two other relevant papers:

Familles et fonctions additives dénsembles abstraits by Maurice Frechet

and

Une série de Fourier-Lebesgue divergent presque partout by Andrei Nikolayevich Kolomogoroff

Steinhaus' 1923 paper followed ten years of his work on triginometric series which continued into 1930s.

J. London Math. Soc. (1929): trigonometric (not Fourier) series , coefficients $\rightarrow 0$, but the series diverges everywhere.

Return to 1912 paper (coefficients not pretty).

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S[x_, n_] := Sum[Cos[k*(t - Log[Log[k]])]/Log[k],
                 {k, 2, n}],  n=2, 25, 100, 1000
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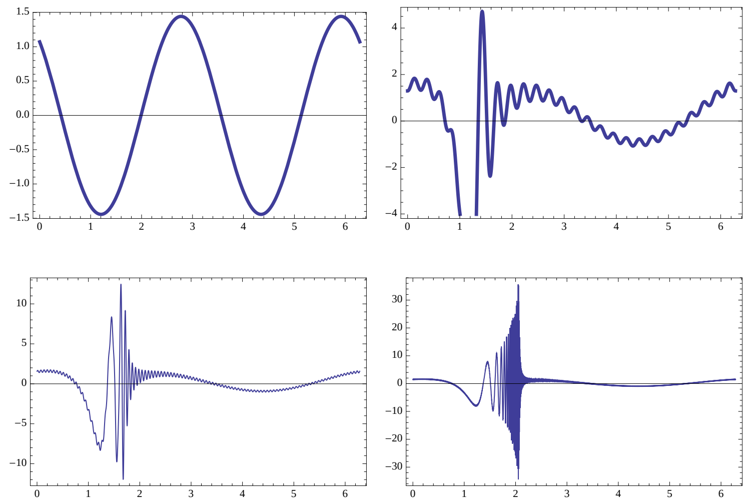


Steinhaus 1923 DOES cite:

Les probabilités dénombrables et leur appplications arithmetiques
Rend. del Circolo Mat. di Palermo (1909)
by Emile Borel

Die Axiome der Wahrscheninlichkeitrechnung
Göttingen PhD (1907)
by Ugo Broggi

Einige Satze über Reihen von algemeinen Orthogonalfunktionen
Math. Ann. (1922)
by Hans Rademacher

Łomnicki studied with Marian Smoluchowski in Lwów who was there from 1901 to 1913, but his approach was completely axiomatic:

1923 paper covered two cases: finite and discrete, and absolutely continuous where the laguage is that of PDFs

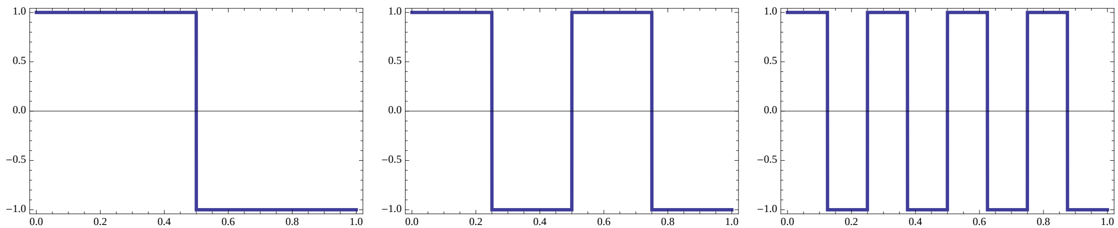
His interest in the abstract approach continued , culminating in a paper with Stanisław Ulam Sur la theorié de la mésure dans les espaces combinatoires et son application au calcul des probabilités
Fund. Math. 23 (1934)

First formal association of independence with product measure (announced at 1932 International Congress in Zurich, one year before Kolmogorov's Grundbegriffe)

Steinhaus, in a sense, fills in the gap in Łomnicki's paper by marrying probability and to an extension of his previous interest from trigonometric to orthogonal series

Models "heads or tails" as a sequence of orthogonal Rademacher functions

$$r_n(x) = \mathrm{sgn}[\sin(2^n x)], \quad x \in [0,1], \qquad n = 0, 1, 2, \ldots,$$



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For Steinhaus, the "measure" in the title of HS 1923 means Lebesgue measure on [0, 1. He will never waver in thinking that it is all that's needed.

You need orthogonal $\sim$ U[0,1]? No problem, says HS 1930: take the binary expansion,

$$\omega = \sum_{n=1}^{\infty} \beta_n(\omega) 2^{-n}, \quad \omega \in [0,1], \quad \beta_n(\omega) = 0, 1,$$

and define

$$\omega_j(\omega) = \sum_{n \in N} \beta_{m(n,j)}(\omega) 2^{-n},$$

where m is a 1-1 mapping from $\mathbf{N}^2$ into $\mathbf{N}$.

Proves SLLN and CLT, also for the binomial case. His basic tool: recently proved theorem by Rademacher:

$$\sum_n c_n^2 < \infty \Rightarrow \sum_n c_n r_n(x) \quad \text{converges a.e. (a.s.)}$$

Essentially he constructed a product measure.

However, HS did not recognize that Rademacher's were independent!!! Mark Kac liked to joke that HS failed to recognize that

$$\frac{1}{2^n} = \frac{1}{2} \cdot \cdots \cdot \frac{1}{2}$$

n times. Had he done so, he could have proved the converse the "only if" part using an early version of Borel-Cantelli Lemma .

Situation similar in SM 1930 where HS proves

$$\sum_n c_n^2 < \infty \Rightarrow \sum c_n e^{2i\omega_n(\omega)} \quad \text{converges a.e.}$$

(one way!!!) without noticing the independence (then Komogorov proves his 3 series criterion) but this direction culminated in definitive monograph with Stefan Kaczmarz on orthogonal series (1936)

Über die Wahrscheinilichkeit dafür, dass de Konvergenzkreis einer Potenzreihe ihre natürlich Grenze ist
Math. Zeitschrift (1930)

If $c_n > 0$, $0 < \limsup c_n^{1/n} < \infty$, and ω_n are (iid) $\sim U[0,1]$, then

$$F(z) = \sum_{n=0}^{\infty} c_n e^{2\pi i \omega_n(\omega)} z^n$$

has, a.e., its circle of convergence as a natural boundary (i.e. it is singular at every point of its circle of convergence); elegant extension by CRN in SM 1953.

Jean Pierre Kahane's 1968 influential Some Random Series of Functions takes this results as its starting point. New areas of harmonic and functional analysis.

Our own 1992 book (Stanisław Kwapień, and WAW) on random series and stochastic integrals, essentially, builds on Steinhaus' tradition.

After publication of Kolmogorov's Grundbegriffe in 1933, and Kaczmarz- Steinhaus (1936) , HS had an epiphany about the concept of independence.

In the next 17 year he published 10 papers , all in SM, entitled Sur les fonctioons independantes, I- X, most with Mark Kac, some single-authored, and one with CRN . The last appeared in 1953.

Has endeavored to show that the whole "hoopla" around the general abstract measure-theoretic treatment of probability is NOT necessary.

Two observation on the independent functions series:

1) Practical foresight: that's how simulations are done now for all stochastic phenomena:

Ulam-Teller-Metropolis' Monte-Carlo

2) Philosophical insight:

When is Random–Random?

(Mark Kac used to give talks with this title)

Take an example (slightly generalized and modernized from paper III in the series)

Consider linearly independent (over rationals) λ_k, such that $\#\{k : \lambda_k < \lambda, k = 1, \ldots n\}/n \to A(\lambda)$. Let

$$x_n(t) = \sqrt{2} \sum_{k=1}^{n} \cos(\lambda_k t)/\sqrt{n}$$

Then, as $n \to \infty$, the joint distribution of $(x_n(t + \tau_1), \ldots x_n(t + \tau_n))$ converges to a multivariate Gaussian distribution with cov matrix $\rho(\tau_i - \tau_j)$, with

$$\rho(\tau) = \int_0^\infty \cos \lambda\tau dA(\lambda).$$

Nothing random about it, yet, asymptotically, $x_n(t)$ cannot be distinguished from a typical sample path of a stochastic stationary Gaussian process $X(t)$ with the integrated power spectrum $A(\lambda)$.

In the same spirit: Effective processes in the sense of H. Steinhaus, SM (1958) by Kazimierz Urbanik ($f^*(I)$ denotes an increment over I)

$$|E|_R = \lim_{T\to\infty} \frac{1}{T} |E \cap \{t: 0 \leqslant t \leqslant T\}|$$

We say that $f(t)$ is *an effective process with independent increments* if for every integer k and for every system of disjoint intervals $I_1, I_2, \ldots, I_k$ the system of functions $g_j(t) = f^*(I_j + t)$ $(j = 1, 2, \ldots, k)$ is relatively measurable,

(1) $$\left|\bigcap_{j=1}^{k} \{t: f^*(I_j + t) < x_j\}\right|_R = \prod_{j=1}^{k} |\{t: f^*(I_j + t) < x_j\}|_R$$

for each $x_1, x_2, \ldots, x_k$ and

(2) $$D_I(x) = |\{t: f^*(I + t) < x\}|_R$$

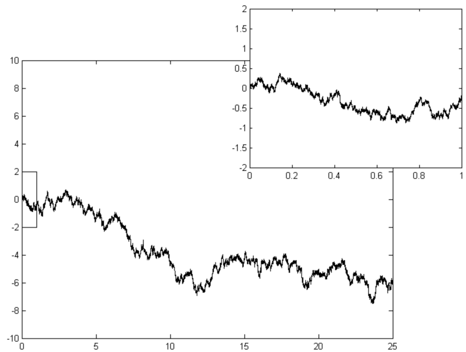
for every interval I is a distribution function, i. e. is a monotone non-decreasing function continuous on the left, with $D_I(-\infty) = 0$, $D_I(\infty) = 1$. (This notion has been proposed by H. Steinhaus).

No effective example of such functions. But

THEOREM. *Let $f(\omega, t)$ be a measurable separable homogeneous stochastic process with independent increments. Then almost all realizations $f(\omega_0, t)$ are effective processes with independent increments. Moreover, for every interval I and for every real number x the equality*

(3) $$|\{t: f^*(\omega_0, I+t) < x\}|_R = P(f^*(\omega, I) < x)$$

is true.



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The last hurray (in theoretical probability):
Poissonsche Folgen
Math. Zeitschrift (1959)
by Hugo Steinhaus and Kazimierz Urbanik

Signal sequence $s_1 < s_2 < \dots$ is called Poisson if $\forall$ disjoint $I_1, I_2, \dots, I_k$, and $\forall$ integers $m_1, m_2, \dots, m_k > 0$

$$|B(I_1, \dots, I_k; m_1, \dots, m_k)|_R = \prod_{r=1}^{k} e^{|I_r|} \frac{|I_r|^{m_r}}{m_r!}$$

where B is the set of t's for which each interval $t + I_r$ contains exactly m_r terms of $\{s_n\}$.

Heads or Tails Forever : Orzeł czy Reszka, Nasz Księgarnia, (1953), 38 pp. Steinhaus' language(s)...

Who is who? This is a test to check if you paid attention. List of drama characters (not in order of appearance but in an appropriately random order):