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Multidimensional Lévy walk and its scaling limits

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Abstract. In this paper we obtain the scaling limit of multidimensional Lévy walk and describe the detailed structure of the limiting process. It occurs that the scaling limit is a subordinated α -stable Lévy motion with the parent process and subordinator being strongly dependent processes. The corresponding Langevin picture is derived. We also introduce a useful method of simulating Lévy walks with predefined spectral measure, which controls the direction of each jump. Our approach can be applied in the analysis of real-life data – we are able to recover the spectral measure from the data and obtain the full characterization of Lévy walk. We also give examples of some useful spectral measures, which cover large class of possible scenarios in the modeling of real-life phenomena.

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1. Introduction

Continuous-time random walk (CTRW), which was introduced to physics by Scher and Montroll [1] as a model of charge carrier transport in amorphous semiconductors, is now a well established mathematical model with variety of applications. It is used in the modeling of: subsurface tracer dispersion [2], electron transfer [3], behavior of dynamical systems [4], self-diffusion in micelles systems [5], transport of light in certain optical materials [6], to name only few.

The classical uncoupled CTRW corresponds to the stochastic scheme, in which every step of random length occurs after random waiting time. It is also assumed that successive jumps and waiting times are independent from the previous ones and from each other. By introducing heavy-tailed jumps and strong dependence between jumps and corresponding waiting times we arrive at the class of Lévy walks (LW). In the standard LW scheme the length of the waiting time is equal (up to a multiplicative constant) to the length of the corresponding jump, which results in constant velocity and finite mean-square displacement of the walker.

Properties of LWs were analyzed in many papers (see eg. [7, 8]). However, most of the obtained results deal with the asymptotic behavior of the mean square displacement and the propagator of LWs. Here we concentrate on the asymptotic properties of the whole trajectories of LWs, which is the main departure from previous studies. In recent papers [9–11] the general problem of weak convergence of trajectories of coupled CTRWs was examined. As LW is a special case of CTRW with strong spatio-temporal coupling, we apply the above mentioned results to investigate convergence of the whole trajectories of multidimensional LW, cf. Fig. 1. In this paper we analyze various properties of the LW limit. It turns out that this limit can be represented as the α -stable Lévy motion subordinated to the inverse α -stable subordinator. As a consequence of strong spatio-temporal coupling in the underlying LW, these two processes are strongly dependent.

LWs have been found as excellent models in the description of various phenomena, such as anomalous diffusion in complex systems [12–19], through human travel [20, 21] and epidemic spreading [22–24], to the foraging patterns of micro-organisms and animals [25–32]. Thus, it is of great importance to develop practical methods, which can be further applied in the statistical analysis of the above mentioned physical and biological systems. Here, we propose methods of approximating the scaling limit of LW. These methods, together with the introduced procedure of recovering statistical properties from the data, give us helpful tools for the analysis of large variety of phenomena observed in nature.

This paper is structured in the following way: in Section 2 we develop the limit theory of multivariate LW and present the Langevin representation of the scaling limit. In Section 3 we propose an alternative construction of multidimensional LW and describe the algorithm of its simulation. We also give analytical description of various spectral measures controlling the jump direction of LW, which can be applied to model real-life phenomena. In Appendices A-C one can find basic facts about d -dimensional α -stable

distributions, as well as the proofs of the main results.

2. Multidimensional Lévy walk and its scaling limit

Let us begin with recalling the notion of CTRW. It is a process, in which consecutive jumps are separated by time intervals of random lengths. A d -dimensional CTRW process with $d \geq 1$ is generated by the sequence of $d + 1$ -dimensional random vectors $\{(\mathbf{X}_i, T_i)\}_{i \geq 1}$, where the positive random variables T_i represent waiting times between consecutive jumps given by d -dimensional random vectors $\mathbf{X}_i$, $i = 1, 2, \dots$. We introduce the counting process $N(t)$

$$N(t) \stackrel{\text{def}}{=} \max\{n \geq 0 : T_1 + T_2 + \dots + T_n \leq t\}, \quad t \geq 0, \quad (1)$$

which counts the number of jumps up to time t , and define the CTRW process

$$\mathbf{R}(t) \stackrel{\text{def}}{=} \sum_{i=1}^{N(t)} \mathbf{X}_i, \quad t \geq 0. \quad (2)$$

The process $\mathbf{R}(t)$ describes position of the walker starting its motion at the origin and following the so-called "wait - jump" pattern, i.e. for each $i \geq 1$ the walker remains in its previous position for time T_i before performing jump $\mathbf{X}_i$.

In this section we analyze the d -dimensional LW, which is a special kind of CTRW generated by the sequence $\{(\mathbf{X}_i, T_i)\}_{i \geq 1}$ satisfying the following conditions:

- (A) $\{\mathbf{X}_i\}_{i \geq 1}$ is a sequence of independent and identically distributed (IID) d -dimensional random vectors, such that the distribution of $\mathbf{X}_1$ belongs to the domain of attraction of some d -dimensional stable law $\mathbf{S}_\alpha(\mathbf{\Lambda}, \mathbf{0})$, (Appendix A), with index of stability $0 < \alpha < 1$ and Fourier transform

$$\begin{aligned} \phi_{\mathbf{S}_\alpha(\mathbf{\Lambda}, \mathbf{0})}(\mathbf{k}) &= E(\exp\{i \langle \mathbf{k}, \mathbf{S}_\alpha(\mathbf{\Lambda}, \mathbf{0}) \rangle\}) \\ &= \exp \left(\int_{\mathbb{S}^d} |\langle \mathbf{k}, \mathbf{s} \rangle|^\alpha (i \operatorname{sgn}(\langle \mathbf{k}, \mathbf{s} \rangle) \tan(\pi\alpha/2) - 1) \mathbf{\Lambda}(d\mathbf{s}) \right), \quad \mathbf{k} \in \mathbb{R}^d. \end{aligned}$$

Here, $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^d$ [33] and $\mathbf{\Lambda}$ is the so-called spectral measure, see Appendix A for the detailed properties and interpretation of $\mathbf{\Lambda}$;

- (B) $T_i = \|\mathbf{X}_i\|$ for each $i \geq 1$, where $\|\mathbf{X}_i\|$ denotes the length of $\mathbf{X}_i$ (the walker moves with constant velocity).

Condition (A) implies that for some sequence of normalizing constants

$$a_n \stackrel{\text{def}}{=} n^{-1/\alpha} q(n)$$

with q being some function slowly varying at infinity [34], we have the following convergence

$$a_n \sum_{i=1}^{\lfloor nt \rfloor} \mathbf{X}_i \Rightarrow \mathbf{L}_\alpha(t) \quad \text{as } n \rightarrow \infty. \quad (3)$$

Here, $\mathbf{L}_\alpha(t)$ is the d -dimensional α -stable Lévy motion [35] with Fourier transform

$$\phi_{\mathbf{L}_\alpha(t)}(\mathbf{k}) = \exp \left(t \int_{\mathbb{S}^d} |\langle \mathbf{k}, \mathbf{s} \rangle|^\alpha (i \operatorname{sgn}(\langle \mathbf{k}, \mathbf{s} \rangle) \tan(\pi\alpha/2) - 1) \Lambda(d\mathbf{s}) \right), \quad \mathbf{k} \in \mathbb{R}^d, \quad (4)$$

see Appendix A. Symbol " $\Rightarrow$ " used in formula (3) denotes convergence in distribution [36].

From Conditions (A) and (B) it follows that the random variables T_i are IID and belong to the domain of attraction of some one-sided α -stable law. Moreover, the following convergence

$$a_n \sum_{i=1}^{\lfloor nt \rfloor} T_i \Rightarrow S_\alpha(t) \quad \text{as } n \rightarrow \infty \quad (5)$$

holds true with the same normalizing constants a_n as in (3), where $S_\alpha(t)$ is the strictly increasing α -stable subordinator [35] with Fourier transform

$$\begin{aligned} \phi_{S_\alpha(t)}(k) &= E(\exp\{ikS_\alpha(t)\}) \\ &= \exp(t\Lambda(\mathbb{S}^d)|k|^\alpha (i \operatorname{sgn}(k) \tan(\pi\alpha/2) - 1)), \quad k \in \mathbb{R}. \end{aligned} \quad (6)$$

It is important to note that as sequences $\{\mathbf{X}_i\}_{i \geq 1}$ and $\{T_i\}_{i \geq 1}$ are strongly dependent, so are the processes $\mathbf{L}_\alpha(t)$ and $S_\alpha(t)$. Particularly, $S_\alpha(t)$ has jumps in the same instants of time and of the same length as $\mathbf{L}_\alpha(t)$. Proofs of formulas (3) - (6) as well as the description of the dependence structure between $\mathbf{L}_\alpha(t)$ and $S_\alpha(t)$ are given in Appendix B.

The inverse α -stable subordinator $S_\alpha^{-1}(t)$ is defined as [37, 38]

$$S_\alpha^{-1}(t) = \inf\{\tau \geq 0 : S_\alpha(\tau) > t\}.$$

The process $S_\alpha^{-1}(t)$ is the scaling limit of the counting process $N(t)$ generated by the sequence of waiting times $\{T_i\}_{i \geq 1}$. We will see in the following that $S_\alpha^{-1}(t)$ occurs as the internal operational time of the LW limit.

Before stating the main result of this section, we need to introduce the left-continuous version of the process $\mathbf{L}_\alpha(t)$, which is defined as

$$\mathbf{L}_\alpha^-(t) = \lim_{s \nearrow t} \mathbf{L}_\alpha(s). \quad (7)$$

The process $\mathbf{L}_\alpha^-(t)$ has the same marginal distribution as $\mathbf{L}_\alpha(t)$, the only difference between both processes is that the trajectories of $\mathbf{L}_\alpha^-(t)$ are left-continuous with right limits whereas the trajectories of $\mathbf{L}_\alpha(t)$ are right-continuous with left limits. As it will be shown in the theorem below, $\mathbf{L}_\alpha^-(t)$ appears in a natural way as driving process in the continuous-time limit of the LW. The occurrence of the process $\mathbf{L}_\alpha^-(t)$ in the limit of LW is the consequence of the fact that the walker follows the so called 'wait-jump' scenario, i.e. i -th jump is preceded by i -th waiting time. This crucial property makes the mean square displacement of LW finite.

We continue with the main result:

Theorem 1 Let $\mathbf{R}(t)$ be a LW process generated by the sequence $\{(T_i, \mathbf{X}_i)\}_{i \geq 1}$ satisfying conditions (A) and (B) and let $\{a_n\}$ be the sequence of normalizing constants as in (3). Then

$$a_n \mathbf{R}(a_n^{-1}t) \Rightarrow \mathbf{Y}(t) \quad \text{as } n \rightarrow \infty,$$

where $\mathbf{Y}(t)$ is the right-continuous version of the process $\mathbf{L}_\alpha^-(S_\alpha^{-1}(t))$ and processes $\mathbf{L}_\alpha(t)$ and $S_\alpha(t)$ have simultaneous jumps of the same lengths. The exact joint distribution of $\mathbf{L}_\alpha(t)$ and $S_\alpha(t)$ is given by formula (B.7) in Appendix B.

For the proof of Theorem 1 see Appendix B.

Let us note that the limit process $\mathbf{Y}(t)$ is in fact a subordinated process. The parent process, which is the left-continuous version of $\mathbf{L}_\alpha(t)$ corresponds to the heavy tailed jumps in the underlying LW. The occurrence of $\mathbf{L}_\alpha^-(t)$ reflects the fact that the first jump of the walker is preceded by the waiting time, which leads to the finite mean-square displacement (MSD) of $\mathbf{Y}(t)$, see [39, 40]. The inverse subordinator $S_\alpha^{-1}(t)$ originates from the heavy tailed waiting times of $\mathbf{R}(t)$. The dependence structure between the jumps of $\mathbf{L}_\alpha(t)$ and $S_\alpha^{-1}(t)$ match the relationship between jumps and waiting times in the underlying LW.

Figure 1. The exemplary trajectory denoted as $\mathbf{R}(t)$ of one-dimensional LW with jumps being symmetric α -stable IID r.v. with $\alpha = 0.9$. Trajectories $\mathbf{R}(4t)/4$, $\mathbf{R}(16t)/16$, $\mathbf{R}(32t)/32$ illustrate the consecutive approximations of the limiting process $\mathbf{Y}(t)$ derived in Theorem 1.

To illustrate the limiting behavior of the scaled LW process, in Fig. 1 we present exemplary trajectory of LW and three rescaled trajectories approximating the limiting

process $\mathbf{Y}(t)$.

2.1. Langevin picture of $\mathbf{Y}(t)$

Now, we apply Theorem 1 to obtain the Langevin picture corresponding to the scaling limit $\mathbf{Y}(t)$. Using the notation of Fogedby introduced in [42], we obtain the following set of coupled Langevin equations describing position x and time t of LW limit

$$\dot{x}(s) = \Gamma(s), \quad \dot{t}(s) = \eta(s) \quad (8)$$

In the above set of equations, the noises $\Gamma(s)$ and $\eta(s)$ are formally equal to

$$\Gamma(s) = \frac{d\mathbf{L}_\alpha^-(s)}{ds}, \quad \eta(s) = \frac{dS_\alpha(s)}{ds}, \quad (9)$$

where $\mathbf{L}_\alpha^-(s)$ is defined in (7) and $S_\alpha(s)$ is given by (6). We underline that $\Gamma(s)$ and $\eta(s)$ are by no means independent. In fact they are very strongly dependent due to the dependence between processes $\mathbf{L}_\alpha^-(t)$ and $S_\alpha(t)$. Equations (8) are interpreted here as usual Langevin equation in an operational time of the system denoted by s . The first equation in (8) determines heavy tailed jumps of the Lévy walker, whereas the second one determines heavy tailed waiting times. Due to the dependence between noises $\Gamma(s)$ and $\eta(s)$, the length of each jump is equal to the length of the waiting time preceding the jump. Therefore, long jumps of the walker are penalized by long waiting times, which in fact makes the mean square displacement finite.

The relationship between the operational time s and the physical time t is described by the second equation in (8). To solve the system of equations (8), first one finds the solution of equation $\dot{x}(s) = \Gamma(s)$ and gets the Langevin process $x(s)$ acting in operational time s . Next, by solving the second equation in (8) one obtains the process $t(s)$. Consequently, by inverting the process $t(s)$ one obtains the process $s(t)$. Finally, the formal solution of (8) has the subordinate form $\mathbf{Y}(t) = x(s(t))$. The structure of the process $\mathbf{Y}(t)$ reflects exactly the structure of the underlying LW – heavy tailed jumps of the process x follow heavy tailed waiting times introduced by the subordinator s .

Numerical simulation of the trajectories of the solution $\mathbf{Y}(t) = x(s(t))$ involves simulation of the processes $\mathbf{L}_\alpha^-$ and S_α . First one simulates the process $\mathbf{L}_\alpha^-(s)$ to obtain via (8) and (9) the process $x(s)$. Next, one simulates $S_\alpha(s)$. We underline that both processes $\mathbf{L}_\alpha^-$ and S_α must have simultaneous jumps of the same length. The details on their simulation can be found in [35,41]. Consequently, by inverting $S_\alpha(s)$ we obtain the subordinator $s(t)$. Finally, putting together x and s , we get the process $\mathbf{Y}(t) = x(s(t))$. Some additional remarks on simulation and estimation procedure will also be presented in the next section.

It should be underlined that Langevin equations (8) can be extended in a natural way by adding external forces. This will allow to analyze various properties of LWs in external potentials. This problem is the subject of our current studies and the obtained results will be presented in a forthcoming paper.

3. Alternative construction of multidimensional Lévy walk. Simulations

Efficient methods of simulation of LW are of great importance, since they can be applied to approximate the limiting process given in Theorem 1 and to verify properties of LWs using Monte Carlo techniques. Although conditions (A) and (B) in Sec. 2 provide a general method of constructing LWs, they have some drawbacks. For example, one has to generate random vectors following a distribution law that belongs to the domain of attraction of stable distribution with some spectral measure $\mathbf{\Lambda}$. Another problem arises, when one needs to choose appropriate normalizing constants a_n for the scaling limit. In this section we present alternative construction of LWs, which allows us to simulate this kind of processes in easy and efficient way.

Motivated by the above remarks we give alternative construction of the sequence $\{(\mathbf{X}_i, T_i)\}_{i \geq 1}$ generating d -dimensional LW. This construction will allow us to determine a priori the spectral measure of the scaling limit and to calculate explicitly the normalizing constants a_n . Assume that

- (I) waiting times T_i are IID random variables distributed according to some heavy-tailed power law with exponent $\alpha \in (0, 1)$, i.e.

$$P(T_i > x) \sim C_0 x^{-\alpha} \text{ as } x \rightarrow \infty$$

for some constant $C_0 > 0$ (for practical reasons it is best to choose T_i from Pareto or stable distribution);

- (II) jumps $\mathbf{X}_i$ are of the form

$$\mathbf{X}_i = T_i \mathbf{V}_i$$

where $\{\mathbf{V}_i\}_{i \geq 1}$ is a sequence of IID random unit vectors in $\mathbb{S}^d$ with distribution $\mathbf{\Lambda}$, independent of sequence $\{T_i\}_{i \geq 1}$ and such that the distribution $\mathbf{V}_1$ it is not degenerated.

Unit vectors $\mathbf{V}_i$ govern the direction of jumps $\mathbf{X}_i$, whereas T_i is the length of jump $\mathbf{X}_i$, $\|\mathbf{X}_i\| = T_i$. In Appendix C we show that conditions (I) and (II) imply conditions (A) and (B) from Sec. 2. Moreover, convergences (3) and (5) holds with normalizing constants given explicitly

$$a_n = (C_0 \Gamma(1 - \alpha) \cos(\pi\alpha/2))^{-1/\alpha} n^{-1/\alpha}. \quad (10)$$

Therefore, the process $\mathbf{L}_\alpha(t)$ has the Fourier transform given by (4) with spectral measure $\mathbf{\Lambda}$ equal to the distribution of the unit vector $\mathbf{V}_1$. Moreover, the Fourier transform of process $S_\alpha(t)$ is of the form

$$\phi_{S_\alpha(t)}(k) = \exp(t|k|^\alpha (i \operatorname{sgn}(k) \tan(\pi\alpha/2) - 1)), \quad k \in \mathbb{R}. \quad (11)$$

In particular case, when the jumps $\mathbf{X}_i = T_i \mathbf{V}_i$ fulfills condition (II) with waiting times T_i following Pareto distribution, then such waiting times and jumps imply conditions (A) and (B), for details see [43].

As a conclusion from the above remarks and Theorem 1 we obtain the following result.

Theorem 2 Let $\mathbf{R}(t)$ be a LW process generated by the sequence $\{(\mathbf{X}_i, T_i)\}_{i \geq 1}$ satisfying conditions (I) and (II). Then

$$\frac{\mathbf{R}(nt)}{n} \Rightarrow \mathbf{Y}(t)$$

where $\mathbf{Y}(t)$ is the right-continuous version of the process $\mathbf{L}_\alpha^-(S_\alpha^{-1}(t))$. Processes $S_\alpha(t)$ and $\mathbf{L}_\alpha(t)$ have simultaneous jumps of the same lengths and their Fourier transforms are given by (11) and (4), respectively. The spectral measure Λ of $\mathbf{L}_\alpha(t)$ is equal to the distribution of the unit vector $\mathbf{V}_1$. Moreover, the exact joint distribution of $\mathbf{L}_\alpha(t)$ and $S_\alpha(t)$ is given by formula (B.7) in Appendix B.

For proof of Theorem 2 see Appendix C.

Let us note that the directional behavior of LW defined by (I) and (II) is determined by the distribution of the unit vector $\mathbf{V}_1$. In the modeling of multidimensional anomalous diffusion problems [44, 45] the distribution of $\mathbf{V}_1$ is called a mixing measure.

Conditions (I) and (II) can be used to introduce numerical procedure of approximation of LWs limiting processes with spatial characteristics given by the spectral measure Λ . The procedure can be summarized in the following steps:

1. simulate an IID sample $\{T_i\}_{i=1}^N$ of LWs' waiting times following Pareto law by taking

$$T_i = (U_i)^{-1/\alpha} \quad \text{for } i = 1, \dots, N,$$

where $\{U_i\}_{i=1}^N$ is a IID sample of random variables uniformly distributed over interval $(0, 1)$,

2. simulate an IID sample $\{\mathbf{V}_i\}_{i=1}^N$ of unit vectors describing the d -dimensional directions of LWs' jumps (see Examples 1-3 and Appendix A for some possible distributions of $\mathbf{V}_i$),
3. put

$$\mathbf{X}_i = T_i \mathbf{V}_i \quad \text{for } i = 1, \dots, N,$$

to construct LWs' jumps,

4. fix large $n \in \mathbb{N}$ and put

$$\tilde{\mathbf{R}}_n(t) = \frac{\mathbf{R}(nt)}{n},$$

where $\mathbf{R}(t)$ is the LW constructed from the above generated sequence $\{(\mathbf{X}_i, T_i)\}_{i=1}^N$. For large values of n , the process $\tilde{\mathbf{R}}_n(t)$ is a good approximation of the scaling limit $\mathbf{Y}(t)$.

It should be underlined that taking larger n (step 4) leads to more accurate approximation of LW scaling limit process. However, simulation of larger samples of waiting times and jumps (steps 1-3) is required to generate longer paths of the approximating process $\tilde{\mathbf{R}}_n(t)$.

The spatial behavior of the obtained LW trajectory is determined by the distribution of $\mathbf{V}_1$, which is related to spectral measure Λ of the limit process $\mathbf{Y}(t)$ by the following equation

$$\mathbf{P}(\mathbf{V}_1 \in \mathbf{D}) = \Lambda(\mathbf{D})$$

for any Borel set $\mathbf{D} \in \mathcal{B}(\mathbb{R}^d)$. Below, we present three examples of LWs with different distributions of $\mathbf{V}_1$, which can be useful in the analysis of empirical data.

Figure 2. The exemplary trajectories (panels a-c) and the corresponding estimated spectral measures (panels d-f) of the limiting processes originating from two-dimensional LW with $\alpha = 0.98$ and $\mathbf{V}_i$ distributed according to: a) uniform distribution, b) von Mises distribution with parameters $\mu = \pi/4$ and $\kappa^{-1} = 0.001$, c) the mixture of two von Mises distributions with parameters $\mu_1 = \pi/2$ and $\kappa_1^{-1} = 0.001$ and $\mu_2 = 3\pi/2$ and $\kappa_2^{-1} = 0.001$ and $p = 0.5$. For the details of spectral measures' estimation, see Sec. 3.

Example 1 *Uniform distribution of $\mathbf{V}_i$*

Let us consider two-dimensional LW, where all directions of jumps are equally likely. This kind of LW can be constructed using IID sequence of waiting times $\{T_i\}_{i \geq 1}$ fulfilling condition (I) and IID sequence of unit vectors $\{\mathbf{V}_i\}_{i \geq 1}$ uniformly distributed over unit circle fulfilling condition (II), i.e. we assume that $\mathbf{V}_i = (\cos U_i, \sin U_i)$, where $\{U_i\}_{i \geq 1}$ is a sequence of IID random variables with uniform distribution over interval $[0, 2\pi)$ independent of $\{T_i\}_{i \geq 1}$. Similar construction was considered in [46] and is often used in the modeling of non-oriented two-dimensional foraging strategies [25–32]. The above described choice of jump direction $\mathbf{V}_i$ results in uniform spectral measure of the limiting process $\mathbf{L}_\alpha(t)$ (details can be found in Appendix C). An exemplary trajectory of the limiting process and the spectral measure of the increments of the limiting process can be found in Fig. 2 in panels a) and d), respectively.

Example 2 *Unimodal circular distribution - von Mises circular distribution of $\mathbf{V}_i$ [48–52]*

Let us consider two-dimensional LW, where the distribution of waiting times is the same as in Example 1 and mixing measure $\mathbf{V}_1$ in (II) is described by von Mises distribution with parameters μ and κ . More precisely, we assume that $\mathbf{V}_i = (\cos(U_i), \sin(U_i))$, where $\{U_i\}_{i \geq 1}$ is a sequence of IID random variables with von Mises distribution over interval $[0, 2\pi)$ independent of $\{T_i\}_{i \geq 1}$ fulfilling condition (I). The probability density function of random variable U_i is the following:

$$f(u) = \frac{\exp(\kappa \cos(u - \mu))}{2\pi I_0(\kappa)}, \quad (12)$$

where I_0 is the modified Bessel function of order 0. Using this kind of mixing measure in LW scheme leads the the limiting processes, which describe anomalous diffusion with preferred orientation. The mean-preferred orientation is given by parameter μ and the concentration of angular directions is controlled by parameter κ . The von Mises distribution closely approximates wrapped normal distribution and is presented here since it is the most tractable non-uniform circular distribution, see [51, 52].

Example 3 *Bimodal circular distribution - mixture of two von Mises circular distributions of $\mathbf{V}_i$ [48–52]*

Let us consider two-dimensional LW, where the distribution of mixing measure $\mathbf{V}_1$ in (II) is given by the mixture of two von Mises distributions. Therefore, for unit vector $\mathbf{V}_i = (\cos U_i, \sin U_i)$, the random variable U_i is described by the following probability density function:

$$f(u) = p \frac{\exp(\kappa_1 \cos(u - \mu_1))}{2\pi I_0(\kappa_1)} + (1 - p) \frac{\exp(\kappa_2 \cos(u - \mu_2))}{2\pi I_0(\kappa_2)}, \quad (13)$$

where I_0 is the modified Bessel function of order 0 and $p \in [0, 1]$ is the weight of first von Mises distribution in the mixture. Using this kind of mixing measure for LW scheme leads the limiting processes, which describe stochastic motion with preferred orientation

(anomalous diffusion in potential well or foraging strategy of animals in some external forces like wind, sea current).

In Fig. 2, in panels a)-c) we present the exemplary trajectories of the limiting process from Theorem 2. To have clear insight into the spatial behavior of the processes, each of the trajectory is presented on the area of the same size. In panels d)-f), there are estimates of the spectral measures corresponding to the trajectories from panels a)-c), respectively. Specifically, the estimated spectral measures are the discrete approximates obtained via the projection method proposed by Nolan, Panorska and McCulloch [53]. For two-dimensional processes the spectral measure is concentrated on unit circle. The length of the bins at particular angle of the unit circle in Fig. 2 in panels d)-f) represents the weight of the spectral measure mass concentrated on unit circle at that angle. In our numerical experiment the estimation of spectral measure was derived using the grid of 2^6 points equally distributed over circle. We can see that the spectral measure in panel d) is uniform. In panels e) and f) we can see two examples of non-uniform spectral measure: uni- and bimodal spectral measures, respectively. They originate from von Mises distribution and from mixture of two von Mises distributions describing unit vectors $\mathbf{V}_i$ defined in condition (II). The results presented in Fig. 2 for simulated data confirm that the method of estimation of spectral measure is very efficient and can be applied for the analysis of real-life data.

4. Summary and conclusions

In this paper the asymptotic behavior of LW was investigated. The main result is the derivation of the scaling limit of multidimensional LW. We have found that the limiting process is a subordinated process, for which the parent process and subordinator are strongly dependent. This property originates from the dependence between waiting times and jumps in the underlying LW. The main result was further applied to introduce the system of Langevin equations describing LW dynamics. The derived explicit formulas for continuous-time limits of multidimensional LWs give insight into the internal structure of the trajectories of the process and allow to analyze their properties.

We have proposed methods of simulation of the limiting process with predefined spectral measure of the parent process. We have visualized the effects of our methods on simulated data. These methods can be further applied in the modeling and statistical analysis of real-life phenomena and also in the estimation of the spectral measure governing the direction of jumps.

Appendix A.

The d -dimensional α -stable random vector arises in a natural way as a limit in distribution of properly scaled and shifted IID d -dimensional heavy-tailed random vectors [54]. Namely, for IID sequence $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n \in \mathbb{R}^d$ of random vectors there exists sequences $a_n \in \mathbb{R}$ and $b_n \in \mathbb{R}^d$ and non-degenerated random vector $\mathbf{X} \in \mathbb{R}^d$ such

that

$$a_n(\mathbf{X}_1 + \mathbf{X}_2 + \dots + \mathbf{X}_n - b_n) \xrightarrow{d} \mathbf{X} \quad \text{as } n \rightarrow \infty$$

if and only if the random vector $\mathbf{X} \in \mathbb{R}^d$ is an d -dimensional α -stable random vector. If the above relation holds we say that random vectors $\mathbf{X}_i$ belong to the domain of attraction of d -dimensional α -stable law.

Neither the cumulative density function nor the probability density function of multidimensional α -stable random vector is known in explicit trivial form. Therefore, the Fourier transform is used for practical reasons. Namely, the d -dimensional α -stable random vector $\mathbf{X}$ with $\alpha \in (0, 2)$ has the following Fourier transform:

$$\phi_{\mathbf{X}}(\mathbf{k}) = \exp\left(\int_{\mathbb{S}^d} |\langle \mathbf{k}, \mathbf{s} \rangle|^\alpha \psi_\alpha(\langle \mathbf{k}, \mathbf{s} \rangle) \Lambda(d\mathbf{s}) + i\langle \mathbf{k}, \mathbf{m} \rangle\right) \quad \text{for any } \mathbf{k} \in \mathbb{R}^d, \quad (\text{A.1})$$

where the function $\psi_\alpha(x)$ is given by

$$\psi_\alpha(x) = \begin{cases} i \operatorname{sgn}(x) \tan(\pi\alpha/2) - 1 & \text{for } \alpha \neq 1, \\ -2i \operatorname{sgn}(x) \ln|x|/\pi - 1 & \text{for } \alpha = 1. \end{cases}$$

Here, Λ is the so-called spectral measure of $\mathbf{X}$. It is a finite measure concentrated over the unit sphere $\mathbb{S}^d$, $\mathbf{m}$ is a vector in $\mathbb{R}^d$ and $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{R}^d$ [33]. The triplet $(\alpha, \Lambda, \mathbf{m})$ determines uniquely α -stable random vector $\mathbf{X}$.

The notation $\mathbf{X} \sim \mathbf{S}_\alpha(\Lambda, \mathbf{m})$ means that $\mathbf{X}$ is the α -stable random vector with spectral measure Λ and shift parameter $\mathbf{m}$. The vector $\mathbf{X}$ is called symmetric α -stable if it is stable and satisfies relation $P(\mathbf{X} \in \mathbf{A}) = P(-\mathbf{X} \in \mathbf{A})$ for any Borel set $\mathbf{A} \in \mathcal{B}(\mathbb{R}^d)$.

Some remarks explaining the role of parameters α , Λ and $\mathbf{m}$ of α -stable random vector are in order. The index of stability α of $\mathbf{S}_\alpha(\Lambda, \mathbf{m})$ determines its heavy-tailed behavior, namely for any X being marginal (or linear combination of marginals) of $\mathbf{S}_\alpha(\Lambda, \mathbf{m})$ we have at least one of the following properties:

$$P(X \geq x) \sim x^{-\alpha} \quad \text{or} \quad P(X \leq -x) \sim x^{-\alpha} \quad \text{as } x \rightarrow \infty.$$

The spectral measure Λ , which is a finite measure concentrated over unit sphere in $\mathbb{R}^d$, determines the dependence between marginals of the α -stable random vector. Below, we present spectral measures of some important α -stable random vectors:

- the spectral measure of d -dimensional α -stable random vector with independent, two-sided marginals is concentrated on the intersection of d -dimensional unit sphere with d -dimensional coordinate axes, see Fig. A1 a);
- the spectral measure of d -dimensional α -stable random vector with all two-sided marginals equal to each other is concentrated in points $(1/\sqrt{d}, \dots, 1/\sqrt{d}) \in \mathbb{R}^d$ and $(-1/\sqrt{d}, \dots, -1/\sqrt{d}) \in \mathbb{R}^d$; see Fig. A1 b);
- the spectral measure of 2-dimensional α -stable vector with second marginal being the absolute value of the first marginal is concentrated at points $(-1/\sqrt{2}, 1/\sqrt{2})$ and $(1/\sqrt{2}, 1/\sqrt{2})$, see Fig. A1 c);

- the spectral measure of 2-dimensional symmetric α -stable vector with uniform, hence also symmetric, spectral measure, see Fig. A1 d).

Vector $\mathbf{m}$ in (A.1) can be interpreted as a shift vector. For more properties of α -stable random vectors see [35, 41].

Figure A1. Visualizations of the spectral measure of 2-dimensional α -stable vector: a) with independent symmetric marginals, b) with symmetric marginals equal to each other, c) with second marginal equal to an absolute value of the first marginal, d) with uniform spectral measure. In all cases $\Lambda(\mathbb{S}^2)=1$. The length of the bins given at particular points of the unit sphere represents the weight of spectral measure mass concentrated at these points. In panel d) the spectral measure is uniform, thus at any two sectors of the unit sphere the mass is equally concentrated.

In one-dimensional case the Fourier transform of α -stable variable X presented in (A.1) reduces to the well know formula:

$$\phi_X(k) = \begin{cases} \exp(ik\mu + |\sigma k|^\alpha (i\beta \operatorname{sgn}(k) \tan(\pi\alpha/2) - 1)) & \text{if } \alpha \neq 1, \\ \exp(ik\mu + |\sigma k|(-2i\beta \operatorname{sgn}(k) \ln(|k|)/\pi - 1)) & \text{if } \alpha = 1. \end{cases} \quad (\text{A.2})$$

involving skewness parameter $\beta \in [-1; 1]$, scale parameter $\sigma > 0$, index of stability $\alpha \in (0; 2)$ and shift parameter μ . For one-dimensional α -stable vector its spectral measure is concentrated on one-dimensional sphere, which in fact consists of two points:

$\{-1\}$ and $\{1\}$. By comparing formula (A.1) with $d = 1$ and (A.2), one can easily derive the following relations between parameters β, σ and spectral measure Λ :

$$\sigma = (\Lambda(1) + \Lambda(-1))^{1/\alpha} \quad \beta = \frac{\Lambda(1) - \Lambda(-1)}{\Lambda(1) + \Lambda(-1)}.$$

When parameter $\beta = 0$ then X is symmetric. On the other hand when $0 < \alpha < 1$ and $\beta = 1$ then we arrive at the class of one-sided positive α -stable random variables. For these and other properties of α -stable distributions, we refer to [35, 41].

Appendix B.

We begin this appendix with proofs of formulas (3) - (6). Next we prove joint functional convergence

$$\left(a_n \sum_{i=1}^{[nt]} \mathbf{X}_i, a_n \sum_{i=1}^{[nt]} T_i \right) \Rightarrow (\mathbf{L}_\alpha(t), S_\alpha(t)) \quad , \quad \forall t \geq 0 \quad (\text{B.1})$$

and discuss properties of limiting process $(\mathbf{L}_\alpha(t), S_\alpha(t))$. Finally we prove Theorem 1.

Observe that convergence (3) and formula (4) follow immediately from condition (A) and Theorem 4.1 [55]. From formula (4), using Theorem 7.3.16 (a) [56] and Lemma 7.3.17 [56] one obtains that Lévy measure of process $\mathbf{L}_\alpha(t)$ is of the form

$$\nu_{\mathbf{L}_\alpha}(dr \times d\theta) = \frac{\alpha}{\Gamma(1-\alpha) \cos\left(\frac{\pi\alpha}{2}\right)} \Lambda(d\theta) r^{-\alpha-1} dr \quad (\text{B.2})$$

where $dr \in (0, \infty)$, $d\theta \in \mathbb{S}^d$. Recall that Lévy measure of any stable distribution is continuous (i.e. it has no atoms). Then applying theorem 3.2.2 [56] to array of random vectors $\{a_n \mathbf{X}_i\}_{n,i \geq 1}$ from condition (A) it follows that

$$nP(a_n \mathbf{X}_1 \in \mathbf{B}) \longrightarrow \nu_{\mathbf{L}_\alpha}(\mathbf{B}) \quad \text{as } n \rightarrow \infty \quad (\text{B.3})$$

for any Borel set $\mathbf{B} \in \mathcal{B}(\mathbb{R}^d \setminus \{\mathbf{0}\})$. Moreover, since stable distribution with $\alpha < 2$ has no Gaussian component we have that

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} n \left[\int_{\|\mathbf{x}\| < \varepsilon} \langle \mathbf{k}, \mathbf{x} \rangle^2 P(a_n \mathbf{X}_1 \in d\mathbf{x}) - \left(\int_{\|\mathbf{x}\| < \varepsilon} \langle \mathbf{k}, \mathbf{x} \rangle P(a_n \mathbf{X}_1 \in d\mathbf{x}) \right)^2 \right] \\ &= \lim_{\varepsilon \rightarrow 0} \liminf_{n \rightarrow \infty} n \left[\int_{\|\mathbf{x}\| < \varepsilon} \langle \mathbf{k}, \mathbf{x} \rangle^2 P(a_n \mathbf{X}_1 \in d\mathbf{x}) - \left(\int_{\|\mathbf{x}\| < \varepsilon} \langle \mathbf{k}, \mathbf{x} \rangle P(a_n \mathbf{X}_1 \in d\mathbf{x}) \right)^2 \right] \\ &= 0 \end{aligned} \quad (\text{B.4})$$

Next we apply Theorem 3.2.2 [56] to array $\{a_n T_i\}_{n,i \geq 1}$ to show that

$$a_n \sum_{i=1}^n T_i \Rightarrow S_\alpha \quad \text{as } n \rightarrow \infty \quad (\text{B.5})$$

where S_α is a random variable with stable distribution $S_\alpha(\sigma, 1, 0)$ whose scale parameter σ satisfies equation $\sigma^\alpha = \Lambda(\mathbb{S}^d)$. Indeed, condition (B) together with (B.2) and (B.3)

yields that

$$\begin{aligned}
n\mathbb{P}(a_n T_1 > r) &= n\mathbb{P}(a_n \|\mathbf{X}_1\| > r) = n\mathbb{P}(a_n \mathbf{X}_1 \in \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\| > r\}) \\
&\longrightarrow \nu_{\mathbf{L}_\alpha}(\{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\| > r\}) = \int_{\mathbb{S}^d} \int_r^\infty \frac{\alpha}{\Gamma(1-\alpha) \cos\left(\frac{\pi\alpha}{2}\right)} \mathbf{\Lambda}(\mathrm{d}\theta) s^{-\alpha-1} \mathrm{d}s \\
&= \left(\Gamma(1-\alpha) \cos\left(\frac{\pi\alpha}{2}\right) \right)^{-1} \mathbf{\Lambda}(\mathbb{S}^d) r^{-\alpha}
\end{aligned} \tag{B.6}$$

for any $r > 0$, thus condition (a) of Theorem 3.2.2 [56] holds with Lévy measure

$$\nu_{S_\alpha}(\mathrm{d}r) \stackrel{\text{def}}{=} \frac{\alpha}{\Gamma(1-\alpha) \cos\left(\frac{\pi\alpha}{2}\right)} \mathbf{\Lambda}(\mathbb{S}^d) r^{-\alpha-1} \mathrm{d}r.$$

Moreover, from (B.4) it follows that condition (b) of Theorem 3.2.2 [56] is satisfied with $Q = 0$, i.e. distribution of S_α has no Gaussian component. Therefore by this theorem convergence (B.5) holds with S_α having infinitely divisible distribution with characteristic exponent

$$\psi_{S_\alpha}(k) = \int_{x>0} (e^{ikx} - 1) \nu_{S_\alpha}(\mathrm{d}x), \quad k \in \mathbb{R}.$$

From Lemma 7.3.7 [56] it follows that S_α has one-sided stable distribution $S_\alpha(\mathbf{\Lambda}(\mathbb{S}^d)^{1/\alpha}, 1, 0)$. Then formula (6) follows from Theorem 7.3.5 [56] and convergence (5) is a consequence of (B.5) and Theorem 4.1 [55].

Now we prove convergence (B.1). Observe that for any Borel sets $\mathbf{B}_1 \in \mathcal{B}(\mathbb{R}^d \setminus \{\mathbf{0}\})$ and $B_2 \in \mathcal{B}((0, \infty))$ we have that

$$\begin{aligned}
\mathbb{P}(a_n \mathbf{X}_1 \in \mathbf{B}_1, a_n T_1 \in B_2) &= \mathbb{P}(a_n \mathbf{X}_1 \in \mathbf{B}_1, a_n \|\mathbf{X}_1\| \in B_2) \\
&= \int_{\mathbf{B}_1} \mathbb{P}(a_n \|\mathbf{X}_1\| \in B_2 \mid a_n \mathbf{X}_1 \in \mathrm{d}\mathbf{x}) \mathbb{P}(a_n \mathbf{X}_1 \in \mathrm{d}\mathbf{x}) \\
&= \int_{\mathbf{B}_1} \mathbb{1}(\|\mathbf{x}\| \in B_2) \mathbb{P}(a_n \mathbf{X}_1 \in \mathrm{d}\mathbf{x})
\end{aligned}$$

where $\mathbb{1}(\|x\| \in B_2)$ equals 1 if condition in the brackets is satisfied and 0 in the opposite case. Hence by (B.3) it follows that

$$n\mathbb{P}(a_n \mathbf{X}_1 \in \mathrm{d}\mathbf{x}, a_n T_1 \in \mathrm{d}s) = \delta_{\|\mathbf{x}\|}(\mathrm{d}s) n\mathbb{P}(a_n \mathbf{X}_1 \in \mathrm{d}\mathbf{x}) \longrightarrow \nu_{(\mathbf{L}_\alpha, S_\alpha)}(\mathrm{d}\mathbf{x} \times \mathrm{d}s) \tag{B.7}$$

where

$$\nu_{(\mathbf{L}_\alpha, S_\alpha)}(\mathrm{d}\mathbf{x} \times \mathrm{d}s) \stackrel{\text{def}}{=} \delta_{\|\mathbf{x}\|}(\mathrm{d}s) \nu_{\mathbf{L}_\alpha}(\mathrm{d}\mathbf{x}) \tag{B.8}$$

and $\delta_{\|\mathbf{x}\|}$ is a probability measure on $(0, \infty)$ with all mass concentrated at point $\|\mathbf{x}\|$. Therefore array $\{(a_n \mathbf{X}_i, a_n T_i)\}_{n,i \geq 1}$ fulfills assumptions of Theorem 3.2.2 [56] with Lévy measure given by (B.8) and with no Gaussian component, so (B.1) holds for fixed $t = 1$. This, by Theorem 4.1 [55] implies weak functional convergence (B.1).

The structure of dependence between processes $\mathbf{L}_\alpha(t)$ and $S_\alpha(t)$ is explained by the joint Lévy measure $\nu_{(\mathbf{L}_\alpha, S_\alpha)}$. For any Borel set $\mathbf{B} \in \mathcal{B}((\mathbb{R}^d \setminus \{\mathbf{0}\}) \times (0, \infty))$ one may interpret quantity $\nu_{(\mathbf{L}_\alpha, S_\alpha)}(\mathbf{B})$ as a measure of intensity of occurrence of jumps taking

values in set $\mathbf{B}$. Using this interpretation it is easy to conclude from (B.8) that processes $\mathbf{L}_\alpha(t)$ and $S_\alpha(t)$ have simultaneous jumps of equal lengths.

We conclude Appendix B with proof of Theorem 1. For this purpose take the array of random vectors $\{(a_n \mathbf{X}_i, a_n T_i)\}$ and define auxiliary sequence of LWs

$$\mathbf{R}_n(t) \stackrel{\text{def}}{=} a_n \sum_{i=1}^{N_n(t)} \mathbf{X}_i \quad \text{where} \quad N_n(t) \stackrel{\text{def}}{=} \max \left\{ k \geq 0 : a_n \sum_{i=1}^k T_i \leq t \right\}.$$

Since convergence (B.1) holds and process $S_\alpha(t)$ has strictly increasing realizations we are able to use Theorem 3.6 [11] and obtain weak functional convergence

$$\mathbf{R}_n(t) \Rightarrow \mathbf{Y}(t). \tag{B.9}$$

Finally observe that

$$N_n(t) = \max \left\{ k \geq 0 : \sum_{i=1}^k T_i \leq a_n^{-1} t \right\} = N(a_n^{-1} t).$$

Hence $\mathbf{R}_n(t) = a_n \mathbf{R}(a_n^{-1} t)$ and from (B.9) follows the thesis of Theorem 1. This completes the proof. $\square$

Appendix C.

Condition (A) together with Theorems 4.5.2 and 4.5.3 [57] yield that convergence (5) holds with normalizing constants given by (10) and in this case process $S_\alpha(t)$ has Fourier transform (11). Moreover, by Theorem 3.2.2 [56] we have that

$$nP(a_n T_1 \in B) \rightarrow \nu_{S_\alpha}(B) \quad \forall B \in \mathcal{B}((0, \infty)) \tag{C.1}$$

and from Theorem 7.3.5 [56] it follows that Lévy measure of $S_\alpha(t)$ is of the form

$$\nu_{S_\alpha}(dx) = \frac{\alpha}{\Gamma(1-\alpha) \cos\left(\frac{\pi\alpha}{2}\right)} x^{-\alpha-1} dx.$$

Next we show that distribution of $\mathbf{X}_1 = T_1 \mathbf{V}_1$ belongs to the domain of attraction of stable law $\mathbf{S}_\alpha(\mathbf{\Lambda}, \mathbf{0})$ with $\mathbf{\Lambda}$ being the distribution of unit vector $\mathbf{V}_1$. Observe that for all sets of the form

$$C(A, D) = \{r\theta : r \in A, \theta \in D\}, \quad A \in \mathcal{B}((0, \infty)), \quad D \in \mathcal{B}(\mathbb{S}^d)$$

convergence (C.1) implies that

$$\begin{aligned} nP(a_n \mathbf{X}_1 \in C(A, D)) &= nP(a_n T_1 \in A, \mathbf{V}_1 \in D) = nP(a_n T_1 \in A)P(\mathbf{V}_1 \in D) \\ &\rightarrow \nu_{S_\alpha}(A)P(\mathbf{V}_1 \in D) = \nu_{S_\alpha}(A)\mathbf{\Lambda}(D). \end{aligned}$$

In view of Example 1.2.22 [56] the above is sufficient in order to show that for the array $\{a_n \mathbf{X}_i\}_{n,i \geq 1}$ condition (a) of Theorem 3.2.2 [56] holds with Lévy measure

$$\nu_{\mathbf{L}_\alpha}(dr \times d\theta) = \nu_{S_\alpha}(dr)\mathbf{\Lambda}(d\theta).$$

Next, by the Cauchy-Schwartz inequality, for any $0 < \varepsilon < 1$ we have that

$$\begin{aligned} 0 &\leq \int_{\|\mathbf{x}\| < \varepsilon} \langle \mathbf{k}, \mathbf{x} \rangle^2 P(a_n \mathbf{X}_1 \in d\mathbf{x}) \leq \int_{\|\mathbf{x}\| < \varepsilon} \|\mathbf{k}\|^2 \|\mathbf{x}\|^2 P(a_n \mathbf{X}_1 \in d\mathbf{x}) \\ &\leq \|\mathbf{k}\|^2 \int_0^\varepsilon r^2 P(a_n T_1 \in dr) \leq \|\mathbf{k}\|^2 \frac{1}{n} \int_0^1 r^2 n P(a_n T_1 \in dr) \rightarrow 0 \quad \text{as } n \rightarrow \infty \end{aligned}$$

since $nP(a_n T_1 \in dr) \rightarrow \nu_{S_\alpha}(dr)$ and $\int_0^1 r^2 \nu_{S_\alpha}(dr) < \infty$. Similarly

$$\begin{aligned} 0 &\leq \left| \int_{\|\mathbf{x}\| < \varepsilon} \langle \mathbf{k}, \mathbf{x} \rangle P(a_n \mathbf{X}_1 \in d\mathbf{x}) \right| \leq \|\mathbf{k}\| \int_{\|\mathbf{x}\| < \varepsilon} \|\mathbf{x}\| P(a_n \mathbf{X}_1 \in d\mathbf{x}) \\ &\leq \|\mathbf{k}\| \frac{1}{n} \int_0^1 r n P(a_n T_1 \in dr) \rightarrow 0 \quad \text{as } n \rightarrow \infty \end{aligned}$$

since $\int_0^1 r \nu_{S_\alpha}(dr) < \infty$. Hence array $\{a_n \mathbf{X}_i\}$ fulfills condition (b) of Theorem 3.2.2 [56]

with $\mathbf{Q} = \mathbf{0}$ and by this theorem we have that $a_n \sum_{i=1}^n \mathbf{X}_i$ converges weakly to the infinitely divisible distribution with Lévy triplet

$$\left[\int_{\|\mathbf{x}\| \neq 0} \frac{\mathbf{x}}{1 + \|\mathbf{x}\|^2} \nu_{\mathbf{L}_\alpha}(d\mathbf{x}), \mathbf{0}, \nu_{\mathbf{L}_\alpha} \right].$$

From Lemma 7.3.17 [56] it follows that this is the Lévy triplet of stable law $\mathbf{S}_\alpha(\mathbf{A}, \mathbf{0})$. Therefore, we have shown that conditions (I) and (II) imply condition (A). It is straightforward that they imply also condition (B).

We conclude Appendix C with proof of Theorem 2. Applying Theorem 1 with normalizing constants a_n given by (10) we have that

$$(Cn)^{-1/\alpha} \mathbf{R}((Cn)^{1/\alpha} t) \Rightarrow \mathbf{Y}(t)$$

where $C = C_0 (\Gamma(1 - \alpha) \cos(\pi\alpha/2))^{-1}$. Since $(Cn)^{1/\alpha} \rightarrow \infty$ as $n \rightarrow \infty$ we replace $(Cn)^{1/\alpha}$ by n and obtain the thesis of Theorem 2. This completes the proof. $\square$

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