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Spectral representation and structure of self-similar processes

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Spectral Representation and Structure of Stable Self-Similar Processes

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Abstract

In this paper we establish a spectral representation of any symmetric stable self-similar process in terms of multiplicative flows and cocycles. A structure of this class of self-similar processes is studied. Applying the Lamperti transformation we obtain a unique decomposition of a symmetric stable self-similar process into three independent parts: mixed fractional motion, harmonizable and evanescent. This decomposition is illustrated by graphical presentation of corresponding kernels of their spectral representations.

1. Introduction

Following the idea from Rosiński ([Ros 1]) for stationary processes we obtain a unique in distribution decomposition of a symmetric α-stable self-similar process $\{X_t\}_{t\in\mathbf{R}_+}$ into three independent parts,

$$X \stackrel{d}{=} X^{(1)} + X^{(2)} + X^{(3)}.$$

Here $\{X_t^{(1)}\}_{t\in\mathbf{R}_+}$ corresponds to a superposition of moving averages in the theory of stationary processes (see [SRMC]). We will call it mixed fractional motion (MFM). This class contains the mixed linear fractional α-stable motion in terminology of Burnecki, Maejima and Weron ([BMW]). The second class $\{X_t^{(2)}\}_{t\in\mathbf{R}_+}$ is harmonizable and $\{X_t^{(3)}\}_{t\in\mathbf{R}_+}$ is called evanescent.

Definition 1.1 *A stochastic process $\{X_t\}_{t\in T}$ is called symmetric α-stable or Lévy $S\alpha S$ or, shortly, $S\alpha S$ process for $\alpha \in (0,2]$ if for every $n \in N$ and any $a_1,\dots,a_n$, $t_1,\dots,t_n \in T$, the random variable $Y = \sum_{i=1}^n a_i X_{t_i}$ has a symmetric stable distribution with index α.*

Definition 1.2 *A family of functions $\{f_t\}_{t\in T} \subset L^\alpha(S,\mathcal{B},\mu)$, where $(S,\mathcal{B},\mu)$ is a standard Lebesgue space, is said to be the kernel of a spectral* *representation of a* $S\alpha S$ *process* $\{X_t\}_{t\in T}$ *if*

$$\{X_t\}_{t\in T} \stackrel{d}{=} \left\{ \int_S f_t(s) M(ds) \right\}_{t\in T}, \tag{1}$$

where M *is an independently scattered random measure on* $\mathcal{B}$ *such that*

$$E \exp\{iuM(A)\} = \exp\{-|u|^\alpha \mu(A)\}, \quad u \in \mathbf{R},$$

for every $A \in \mathcal{B}$ *with* $\mu(A) < \infty$. *A kernel* $\{f_t\}_{t\in T}$ *is said to be minimal if* $\sigma\{f_t/f_u : t, u \in T\} = \mathcal{B}$ *modulo* μ.

Every separable in probability $S\alpha S$ process has a minimal representation (see [Har] and [JW]). Note that the definition of minimality given here is equivalent to the original definition but is easier to formulate (see [Ros 2]); the latter work provides several workable tests for the verification of minimality in concrete cases. We will also consider complex stable processes. In the complex case, f_t are complex valued and M is invariant under rotations.

2. General spectral representation

From now on we will consider processes indexed by $T = \mathbf{R}_+ = (0, \infty)$. A stochastic process $\{X_t\}_{t>0}$ is said to be H-*self-similar* $(H - ss)$ if $\{X_{ct}\}_{t>0} =^d \{c^H X_t\}_{t>0}$, for every $c > 0$. In this section we will characterize the kernel of a spectral representation of a self-similar $S\alpha S$ stochastic process. Without loss of generality we may and do assume that underlying measure space $(S, \mathcal{B}, \mu)$ for the kernel is Borel. A collection $\{\phi_t\}_{t>0}$ of measurable maps from S onto S such that

$$\phi_{t_1 t_2}(s) = \phi_{t_1}(\phi_{t_2}(s)) \tag{2}$$

and $\phi_1(s) = s$ for all $s \in S$ and $t_1, t_2 > 0$ is called a *multiplicative flow*. Such flow is said to be measurable if the map $\mathbf{R}_+ \times S \ni (t, s) \mapsto \phi_t(s) \in S$ is measurable. Given a $\sigma-$finite measure μ on $(S, \mathcal{B})$, $\{\phi_t\}_{t>0}$ is said to be *nonsingular* if $\mu(\phi_t^{-1}(A)) = 0$ if and only if $\mu(A) = 0$ for every $t > 0$ and $A \in \mathcal{B}$.
Let A be a locally compact second countable group. A measurable map $\mathbf{R}_+ \times S \ni (t, s) \to a_t(s) \in A$ is said to be a cocycle for a measurable flow $\{\phi_t\}_{t>0}$ if for every $t_1, t_2 > 0$

$$a_{t_1 t_2}(s) = a_{t_2}(s) a_{t_1}(\phi_{t_2}(s)) \quad \textit{for all } s \in S. \tag{3}$$

Theorem 2.1 *Let $\{f_t\}_{t>0} \subset L^\alpha(S,\mu)$ be the kernel of a measurable minimal spectral representation of a measurable $H-ss$ $S\alpha S$ process $\{X_t\}_{t>0}$. Then there exist a unique modulo μ nonsingular flow $\{\phi_t\}_{t>0}$ on (S,μ) and a cocycle $\{a_t\}_{t>0}$ taking values in $\{-1,1\}$ ($\{|z|=1\}$ in the complex case) such that for each $t>0$*

$$f_t = t^H a_t \Big\{\frac{d\mu\circ\phi_t}{d\mu}\Big\}^{1/\alpha}(f_1\circ\phi_t) \quad \mu-a.e. \tag{4}$$

Proof. Since $t\to f_t$ is minimal, then, for each $c>0$ $\{1/c^H f_{ct}\}_{t>0}$ and $\{f_t\}_{t>0}$ are kernels of minimal representations of the the same $H-ss$ $S\alpha S$ process. Applying Theorem 2.2 in [Ros 1] there exist a one-to-one and onto function $\Phi_c : S\to S$ and a function $h_c : S\to \mathbf{R}-\{0\}$ such that, for each $t>0$,

$$f_{ct} = (c^H)(h_c)(f_t\circ\Phi_c) \quad \mu-a.e., \tag{5}$$

and

$$\frac{d(\mu\circ\Phi_c)}{d\mu} = |h_c|^\alpha, \quad \mu-a.e. \tag{6}$$

Since, for every $t,c_1,c_2>0$, it is true that, $\mu-a.e$

$$f_{c_1c_2t} = (c_2^H)(h_{c_2})(f_{c_1t}\circ\Phi_{c_2}) = (c_2^Hc_1^H)(h_{c_2})(h_{c_1}\circ\Phi_{c_2})(f_t\circ\Phi_{c_1}\circ\Phi_{c_2}) \tag{7}$$

and

$$f_{c_1c_2t} = (c_1^Hc_2^H)(h_{c_1c_2})(f_t\circ\Phi_{c_1c_2}),$$

we infer from Theorem 2.2 in [Ros 1] that, for every $c_1,c_2>0$,

$$h_{c_1c_2} = (h_{c_2})(h_{c_1}\circ\Phi_{c_2}), \quad \mu-a.e., \tag{8}$$

and

$$\Phi_{c_1c_2} = \Phi_{c_1}\circ\Phi_{c_2}, \quad \mu-a.e. \tag{9}$$

In order to conclude the proof it is enough to rewrite the arguments of the proof of Theorem 3.1 in [Ros 1] replacing the additive group $\mathbf{R}$ with the multiplicative $\mathbf{R}_+$. Therefore, $\phi_t=\Phi_t$ is the map and putting $a_t = h_t/|h_t|$ ends the proof. □

Remark It is possible to present another proof of the theorem using the Lamperti transformation defined in the following:

Lemma 2.1 *([Lam]) If $\{Y_t\}_{t\in\mathbf{R}}$ is a stationary process and if for some $H>0$*

$$X_t = t^H Y_{\log t}, \quad for \;\; t>0, \;\; X_0=0, \tag{10}$$

then X_t is H-ss. Conversely, every non-trivial ss-process with $X_0=0$ is obtained in this way from some stationary process Y.

Namely, first we need to see that the Lamperti transformation leading from self-similar to stationary processes preserves the minimality of the spectral representation. To this end it is enough to verify condition (iii) of Theorem 3.8 in [Ros 2] with $F = \{e^{-tH} f_{e^t}\}_{t\in\mathbf{R}}$. It is trivially satisfied as the condition is fulfilled for $F = \{f_t\}_{t\in\mathbf{R}_+}$. Now, taking $Y_t = e^{-tH} X_{e^t}$ we obtain a stationary process which minimal representation is defined by Theorem 3.1 in [Ros 1] in terms of a unique flow and a corresponding cocycle on the additive group $\mathbf{R}$. In order to conclude the proof we apply the reciprocal transformation $X_t = t^H Y_{\log t}$ which leads to the minimal spectral representation of the process X as stated in Theorem 2.1. □

Corollary 2.1 *Since there is a correspondence between self-similar and stationary processes through Lamperti transformation every minimal representation $t \to f_t$ (4) given in terms of a flow ϕ_t and a cocycle a_t defines the kernel of a minimal spectral representation $\{f_t^1\}_{t\in\mathbf{R}}$ of the corresponding stationary process as follows*

$$f_t^1 = a_t^1 \left\{ \frac{d\mu \circ \phi_t^1}{d\mu} \right\}^{1/\alpha} (f_0 \circ \phi_t^1), \quad \mu - a.e. \tag{11}$$

such that

$$\phi_t^1(s) = \phi_{e^t}(s),\ a_t^1(s) = a_{e^t}(s),\ f_0^1(s) = f_1(s) \quad \textit{for all } s \in S \textit{ and } t \in \mathbf{R}.$$

Conversely if (11) is the kernel of a minimal spectral representation of a stationary process then (4) defines the kernel of a minimal representation of an $H-ss$ process in terms of a pair $\{a_t, \phi_t\}_{t>0}$ such that

$$\phi_t(s) = \phi_{\log t}^1(s),\ a_t(s) = a_{\log t}^1(s),\ f_1(s) = f_0^1(s) \quad \textit{for all } s \in S \textit{ and } t > 0.$$

Remark Combining results of Theorem 3.1 in [Ros 1] and Theorem 2.1 we may try to describe classes of transformations leading from self-similar to stationary processes and conversely in the similar way as in Theorems 3.1 and 3.2 in [BMW], which are the following.

Theorem 2.2 *(*[BMW]*) Let $0 < H < \infty$.*

(i) If for some continuous functions θ, ψ : $(0,\infty) \to \mathbf{R}$ and a non-trivial stationary process $\{Y_t\}_{t\in\mathbf{R}}$,

$$X_t = \begin{cases} \theta(t) Y_{\psi(t)}, & \textit{for } t > 0 \\ 0, & \textit{for } t = 0 \end{cases} \tag{12}$$

is $H-ss$, then $\theta(t) = t^H$ and $\psi(t) = a \log t$ for some $a \in \mathbf{R}$.

(ii) If for some continuous functions ζ, η : $\mathbf{R} \to (0,\infty)$ such that η is invertible and for a non-trivial $H-ss$ process $\{X_t\}_{t\in\mathbf{R}}$,

$$Y_t = \zeta(t)X_{\eta(t)}, \quad t \in \mathbf{R},$$

is stationary, then

$$\zeta(t) = e^{-bHt} \ \ and \ \ \eta(t) = e^{bt} \ \ for\ some\ b \in \mathbf{R}.$$

Sketch of the proof. Let us concentrate on (i). We will support the thesis that $\theta = t^H$ and $\psi = a\log t$ using Theorem 3.1 in [Ros 1] and Theorem 2.1 which concern minimal spectral representations of stationary and self-similar processes, respectively. First we notice that any transformation of the form $X_t = \theta(t)Y_{\psi(t)}$ for a non-trivial stationary process Y and functions θ, ψ : $(0,\infty) \to \mathbf{R}$ such that ψ is onto preserves minimality of the spectral representation. It is obvious since $F = \{\theta(t)f^1_{\psi(t)}\}_{t>0}$ satisfies condition (iii) of Theorem 3.8 in [Ros 2] as $\{f^1_t\}_{t\in\mathbf{R}}$ (the spectral representation of process Y) is rigid in $L^\alpha(S,\mu)$. Thus X is $H-ss$ with the spectral representation as follows

$$f_t = \theta(t)a^1_{\psi(t)}\Big\{\frac{d\mu\circ\phi^1_{\psi(t)}}{d\mu}\Big\}^{1/\alpha}(f_0\circ\phi^1_{\psi(t)}) \quad \mu-a.e.$$

Now we use the fact that the process X has a spectral representation defined by (4) and compare them. We immediately obtain that $\theta(t) = t^H$. Furthermore, it is easy to see that the spectral representations are equivalent if

$$\phi^1_{\psi(t_1t_2)} = \phi^1_{\psi(t_1)+\psi(t_2)} \ \ and \ \ \psi(1) = 0.$$

This yields either

$$\psi(t_1t_2) = \psi(t_1) + \psi(t_2) \quad for\ all\ t_1, t_2 > 0 \tag{13}$$

or

$$\psi(t_1t_2) = \psi(t_1) + \psi(t_2) + c \quad for\ some\ t_1, t_2 > 0\ and\ c \neq 0.$$

Since ψ is continuous the latter implies that Y is trivial. The equivalence (13) leads to the statement $\psi(t) = a\log t$ for some real constant a. □

3. Mixed fractional motion

The simplest $H-ss$ $S\alpha S$ process is obtained from a kernel of the form

$$f_t(s) = t^{H-\frac{1}{\alpha}}f\left(\frac{s}{t}\right), \qquad t, s > 0, \tag{14}$$

considered with Lebesgue control measure on $(0,\infty)$, $f \in L^\alpha((0,\infty), Leb)$. A $S\alpha S$ process with such representation will be called a *fractional motion* (FM). A superposition of independent FM processes of type (14) is called a *mixed fractional motion* (MFM).

Definition 3.1 *An* $H-ss$ $S\alpha S$ *process* $\{X_t\}_{t>0}$ *is said to be a MFM if it admits a spectral representation with a kernel* $\{g_t\}_{t>0}$ *defined on* $(W \times (0,\infty), \mathcal{B}_W \otimes \mathcal{B}_{(0,\infty)}, \nu \otimes Leb)$*, for some Borel measure space* $(W, \mathcal{B}_W, \nu)$*, such that*

$$g_t(w,u) = t^{H-\frac{1}{\alpha}} g\left(w, \frac{u}{t}\right), \tag{15}$$

$(w,u) \in W \times (0,\infty), \quad t > 0.$

We will give a few examples of FM and MFM processes. We begin with the simplest one.

Example 3.1 *Let* $0 < \alpha < 2$, $H = \frac{1}{\alpha}$ *and* $\{X\}_{t>0}$ *be a Lévy motion. Then*

$$X_t = \int_0^t M(ds) = \int_0^\infty f(s/t) M(ds),$$

where

$$f(s) = I[0 < s < 1]$$

and M *is* $S\alpha S$ *on* $(0,\infty)$ *with Lebesgue control measure.*

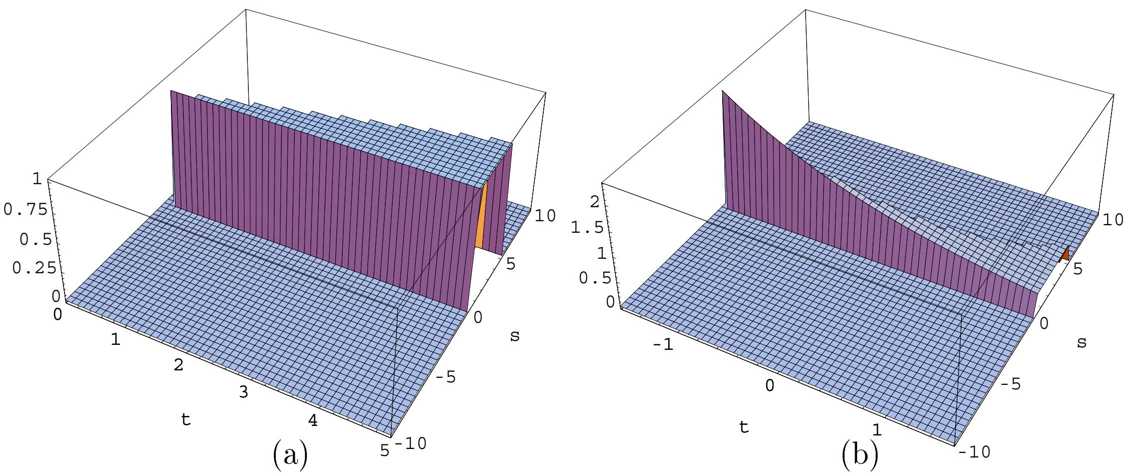

Figure 1: (a) The kernel of the spectral representation of Lévy motion, (b) the kernel of the corresponding stationary process through the Lamperti transformation for $H = 1/1.8$ (i.e. Ornstein-Uhlenbeck process).

Example 3.2 *Let $f \in L^\alpha(\mathbf{R}^d, Leb)$. Let*

$$f_t(s) = t^{H-\frac{d}{\alpha}} f\left(\frac{s}{t}\right), \quad s \in \mathbf{R}^d,\ t > 0,$$

and let M be a $S\alpha S$ random measure on $\mathbf{R}^d$ with Lebesgue control measure. It is easy to check that a $S\alpha S$ process $\{X_t\}_{t>0}$ with such spectral representation is H-ss. We will show that $\{X_t\}_{t>0}$ is a MFM. Indeed, let $W = S_d$ be the unit sphere in $\mathbf{R}^d$ equipped with the uniform probability measure ν and let

$$g(w,u) = (c_d u^{d-1})^{1/\alpha} f(uw), \quad (w,u) \in S_d \times (0,\infty),$$

where $c_d = 2\pi^{d/2}/\Gamma(d/2)$ is the surface area of S_d. Using polar coordinates, we get for every $a_1, \ldots, a_n \in \mathbf{R}$, $t_1, \ldots, t_n > 0$,

$$\int_{\mathbf{R}^d} |\sum a_j f_{t_j}(s)|^\alpha \, ds$$

$$= c_d \int_{S_d} \int_0^\infty |\sum a_j t_j^{H-\frac{d}{\alpha}} f\left(\frac{uw}{t_j}\right)|^\alpha u^{d-1} \, du \nu(dw)$$

$$= \int_{S_d} \int_0^\infty |\sum a_j t_j^{H-\frac{1}{\alpha}} g\left(w, \frac{u}{t_j}\right)|^\alpha \, du \nu(dw),$$

which proves the claim.

Comparing the kernel from the above example with the general form (4) we get that $S = \mathbf{R}^d \setminus \{0\}$, $\phi_t(s) = t^{-1}s$, $f_1(s) = f(s)$, and $\frac{d\mu\circ\phi_t}{d\mu} = t^{-d}$. The following well-known $H-ss$ processes are special cases of Example 3.2.

Example 3.3 *Let $1 < \alpha < 2$ and $H = \frac{1}{\alpha}$. Then a log-fractional motion (cf. [KMV]) $\{X_t\}_{t>0}$ is defined by*

$$X_t = \int_{-\infty}^\infty \log\left|\frac{t-s}{s}\right| M(ds) = \int_{-\infty}^\infty f(s/t) M(ds),$$

where

$$f(s) = \log|1/s - 1|$$

and M is $S\alpha S$ on $\mathbf{R}$ with Lebesgue control measure.

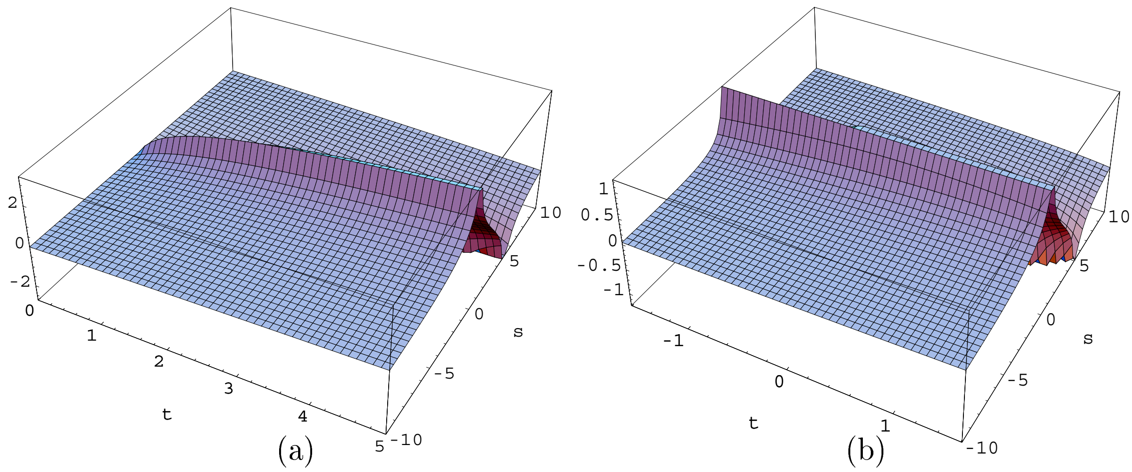

Figure 2: (a) The kernel of the spectral representation of log-fractional motion, (b) the kernel of the corresponding stationary process for $H = 1/1.8$.

Example 3.4 *Let $0 < H < 1$, $0 < \alpha < 2$, $H \neq \frac{1}{\alpha}$. Put $\beta = H - \frac{1}{\alpha}$. Then a linear fractional stable motion (cf. [CMS]) $\{X_t\}_{t>0}$ is defined by*

$$X_t = \int_{-\infty}^{0} p[(t-s)^\beta - (-s)^\beta]\, M(ds) +$$

$$\int_0^\infty \Big(I[0 < s < t][p(t-s)^\beta - qs^\beta] + I[t < s]q[(s-t)^\beta - s^\beta]\Big)\, M(ds)$$

$$= \int_{-\infty}^{\infty} t^\beta f(s/t) M(ds),$$

where

$$f(s) = I[s < 0]p[(1-s)^\beta - (-s)^\beta] +$$

$$I[0 < s < 1][p(1-s)^\beta - qs^\beta] + I[s > 1]q[(s-1)^\beta - s^\beta],$$

and M is $S\alpha S$ on $\mathbf{R}$ with Lebesgue control measure.

Next Theorem shows that the kernel of a spectral representation of any MFM can be defined on $\mathbf{R}^2$ in a canonical way.

Theorem 3.1 *(Canonical representation of a MFM). Let σ be a σ-finite measure on the unit circle S_2 of $\mathbf{R}^2$ and let μ be a measure on $\mathbf{R}^2 \setminus \{0\}$ whose representation in polar coordinates is*

$$\mu(dr, d\theta) = r^{\alpha H - 1} dr\, \sigma(d\theta), \qquad r > 0,\ \theta \in S_2. \tag{16}$$

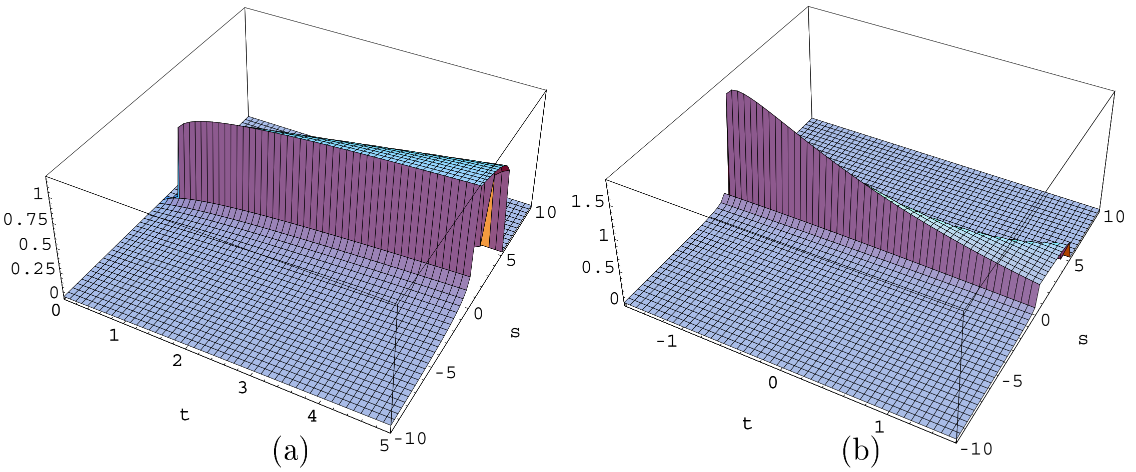

Figure 3: (a) The kernel of the spectral representation of linear fractional stable motion for $H - 1/\alpha = 0.1$, (b) the kernel of the corresponding stationary process for $H = 0.1 + 1/1.8$.

Let $f : \mathbf{R}^2 \setminus \{0\} \mapsto \mathbf{R}$ *(or* $\mathbf{C}$*) be such that*

$$\int_{\mathbf{R}^2\setminus\{0\}} |f(z)|^\alpha \, \mu(dz) < \infty.$$

Then the family of functions $\{f_t\}_{t>0} \subset L^\alpha(\mathbf{R}^2 \setminus \{0\}, \mu)$ *given by*

$$f_t(z) = f(t^{-1}z) \tag{17}$$

is the kernel of a spectral representation of a $S\alpha S$ *process, which is* $H - ss$ *and MFM. Conversely, every MFM admits a (canonical) representation (16)-(17).*

Proof. We are to show only the converse part. Consider a MFM with a representation (15). Since S is a Borel space, S is measurably isomorphic to a Borel subset S_2. Let $\Phi : S \mapsto S_2$ denote this isomorphism and let $\sigma = \nu \circ \Phi^{-1}$. Define a function f on $\mathbf{R}^2 \setminus \{0\}$ as follows

$$f(z) = \begin{cases} g\left(\Phi^{-1}\left(\frac{z}{|z|}\right), |z|\right) |z|^{1/\alpha - H}, & \text{if } \frac{z}{|z|} \in \Phi(S) \\ 0, & \text{otherwise.} \end{cases}$$

Let μ be a measure on $\mathbf{R}^2 \setminus \{0\}$ given by (16). Then

$$\int_{\mathbf{R}^2\setminus\{0\}} |\sum a_j f_{t_j}(z)|^\alpha \, \mu(dz) = \int_{\mathbf{R}^2\setminus\{0\}} |\sum a_j f(t_j^{-1}z)|^\alpha \, \mu(dz)$$

$$= \int_{S_2} \int_0^\infty |\sum a_j f(t_j^{-1} r\theta)|^\alpha \, r^{\alpha H - 1} \, dr \sigma(d\theta)$$

$$= \int_S \int_0^\infty |\sum a_j f(t_j^{-1} r \Phi(s))|^\alpha \, r^{\alpha H - 1} \, dr \nu(ds)$$
$$= \int_S \int_0^\infty |\sum a_j t_j^{H-1/\alpha} g(s, t_j^{-1} r)|^\alpha \, dr \nu(ds),$$

for every $t_1, \ldots, t_n > 0$ and $a_1, \ldots, a_n \in \mathbf{R}\,(\mathbf{C})$. This ends the proof. □

Remark. The Lamperti transformation maps FMs onto moving average processes and MFMs onto mixed moving averages (see [SRMC]). Considering above examples it seems that MFMs appear more naturally than FMs. This is quite opposite to the relation between mixed and the usual moving averages.

It is clear that a stable process may have many spectral representations with different kernels defined on various measure spaces. However, we can identify one property, common to all such representations, which characterizes MFMs.

Theorem 3.2 *Let $\{X_t\}_{t>0}$ be a $S\alpha S$ $H-ss$ process with an arbitrary representation (1). Then X is MFM if and only if*

$$\int_0^\infty t^{-\alpha H - 1} |f_t(s)|^\alpha dt < \infty \quad \mu - a.e. \tag{18}$$

Proof. The condition (18) is equivalent to

$$\int_{-\infty}^\infty e^{-\alpha H t} |f_{e^t}(s)|^\alpha dt < \infty \quad \mu - a.e.$$

By Theorem 2.1 in [Ros 2] and (10) this concludes the proof. □

4. Decomposition of stable self-similar processes

Similarly as in the case of stationary $S\alpha S$ processes, Theorem 2.1 allows one to use ergodic theory ideas in the study of $S\alpha S$ self-similar processes. In particular, the Hopf decomposition of the underlying space S of the spectral representation (4) into invariant parts C and D, such that the flow ϕ_t is conservative on C and dissipative on D, generates a decomposition of $\{X_t\}_{t>0}$ into two independent $S\alpha S$ $H-ss$ processes $\{X_t^C\}_{t>0}$ and $\{X_t^D\}_{t>0}$. We will characterize the latter process.

Theorem 4.1 *$\{X_t^D\}_{t>0}$ is a MFM and one can choose a minimal representation of $\{X_t^D\}_{t>0}$ of the form (15). Furthermore, $\{X_t^D\}_{t>0}$ is a FM if and only if $\{\phi_t\}_{t>0}$ restricted to D is ergodic.*

Proof. Using Corollary 2.1 we infer that the process $\{X_t^D\}_{t>0}$ corresponds, by Lamperti transformation, to a stationary $S\alpha S$ process $\{Y_t\}_{t\in\mathbf{R}}$ generated by a dissipative flow. From Theorem 4.4 in [Ros 1] we get that $\{Y_t\}_{t\in\mathbf{R}}$ is a mixed moving average, implying that $\{X_t^D\}_{t>0}$ is a MFM.

We will now prove the second part of the theorem. Since a moving average representation kernel is minimal (see, e.g, [Ros 2]), (14) is minimal as well. Since f_t in (4) is minimal, then also f_t restricted to D is minimal. By Theorem 3.6 in [Ros 1] we infer that the (multiplicative) flow ϕ_t is equivalent to the flow $\psi_t(s) = t^{-1}s,\ \ t, s > 0$. Since $\{\psi_t\}$ is ergodic, so is $\{\phi_t\}$. Now suppose that $\{\phi_t\}$ is ergodic. By the first part of this theorem, $\{X_t\}$ admits a minimal representation of the form (15) whose flow is given by $\psi_t(w, u) = (w, t^{-1}u)$. Since the latter flow is equivalent to $\{\phi_t\}$ by the foregoing theorem, it must be ergodic which is only possible when ν is a point-mass measure. Thus (15) reduces to (14). □

The class generated by conservative flows consists of harmonizable processes and processes of a third kind (evanescent).

Definition 4.1 *An $H-ss$ process $\{X_t\}_{t>0}$ is said to be* harmonizable *if it admits the representation*

$$\{X_t\}_{t>0} =_d \left\{ \int_{\mathbf{R}} t^{H+is} N(ds) \right\}_{t>0}, \tag{19}$$

where N is a complex-valued rotationally invariant $S\alpha S$ measure with the finite control measure ν on S.

Notice that the representation (19) is minimal and it is generated by an identity flow acting on S with $a_t(s) = t^{is}$ as the corresponding multiplicative cocycle. It is easy to prove the converse:

Proposition 4.1 *Let $\{X_t\}_{t>0}$ be a measurable complex-valued $H-ss$ $S\alpha S$ process generated by an identity flow. Then $\{X_t\}_{t>0}$ is harmonizable.*

Proof. Let

$$S_0 = \{s : a_{t_1 t_2}(s) = a_{t_1}(s) a_{t_2}(s) \ \ for \ \ Leb \otimes Leb \ \ a.a. \ (t_1, t_2)\}.$$

Now it is enough to show that for each $s \in S_0$ there exist a unique $k(s) \in \mathbf{R}$ such that

$$a_t(s) = t^{ik(s)}.$$

To this end we follow the proof of Proposition 5.1 in [Ros 1] and next define a finite measure $\mu_0(ds) = |f(s)|^\alpha \mu(ds)$ on S. Therefore, (19) holds with $\nu = \mu_0 \circ k^{-1}$. □

Theorem 4.2 *Let* $\{f_t\}_{t>0}$ *be the kernel of a minimal spectral representation of the form (4) for a complex-valued* $S\alpha S$ *harmonizable process* $\{X_t\}_{t>0}$. *Then* $\{\phi_t\}_{t>0}$ *is the identity flow and (4) reduces to*

$$f_t(s) = t^{H+is} f(s) \tag{20}$$

Proof. Since (20) follows from the proof of the previous proposition, we only need to show that $\{\phi_t\}_{t>0}$ is the identity flow. However, the representation (19) is minimal and is induced by the identity flow $\psi_t(s) = s$, for all t, s, so that by Theorem 3.6 in [Ros 1], ϕ_t being equivalent to the identity flow must be identity. □

Example 4.1 *Let*

$$\{X_t\}_{t>0} =_d \left\{ \int_{-\infty}^{\infty} t^{H+is} \frac{e^{is}-1}{is} |s|^{-(H-1/2)} M(ds) \right\}_{t>0},$$

where M *is a complex-valued rotationally invariant* $S\alpha S$ *measure. The process* X *corresponds via the Lamperti transformation to the increment process of fractional Gaussian noise (cf. [ST]).*

Remark. There can not be any non-zero real-valued stationary harmonizable process. Using Lamperti transformation, the same statement is valid about real-valued harmonizable self-similar processes. However, the class of real-valued self-similar processes whose spectral represenation is generated by the identity flow is slightly larger. Any process of this class must be of the form $X_t = t^H X_1$ (cf. Proposition 5.2 in [Ros 1]).

Definition 4.2 *A stochastic process whose minimal representation (4) contains a conservative flow without fixed points will be called* evanescent.

This class is not well understood at present. The next theorem is useful to verify whether or not a process is evanescent.

Theorem 4.3 *Let* $\{X_t\}_{t>0}$ *be a* $S\alpha S$ $H-ss$ *process with an arbitrary representation (1). Then* $\{X_t\}_{t>0}$ *is evanescent if and only if*

$$\mu\{s \in S: \int_0^{\infty} t^{-\alpha H-1} |f_t(s)|^{\alpha}\, dt < \infty\} = 0$$

and

$$\mu\{s \in S: f_{t_1 t_2}(s) f_1(s) = f_{t_1}(s) f_{t_2}(s) \text{ for a.a. } t_1, t_2 > 0\} = 0$$

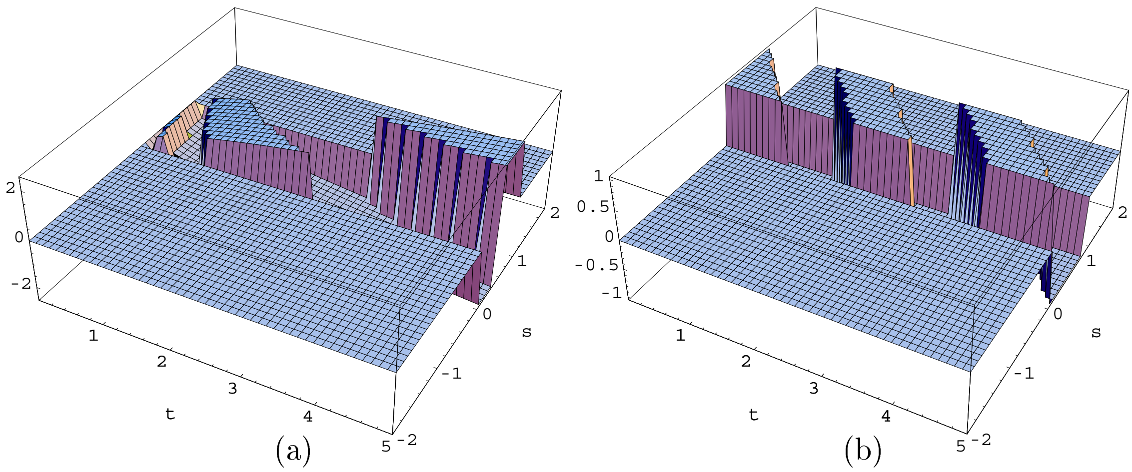

Figure 4: (a) The kernel of the spectral representation of the evanescent process, (b) the kernel of the corresponding stationary process for $H = 1/1.8$.

Proof. It is a direct consequence of results of Section 6 in [Ros 1] combined with Lamperti transformation, Lemma 2.1. □

We will give two examples of evanescent processes.

Example 4.2 *Let*

$$\{X_t\}_{t>0} =_d \left\{ \int_0^1 t^H \cos \pi [\log t + s] M(ds) \right\}_{t>0},$$

where $[x]$ *denotes the largest integer not exceeding* x. *Then* X *does not have a corresponding harmonizable nor mixed moving average component, so provides an example of an evanescent component.*

Example 4.3 *Let* $\{X_t\}_{t>0}$ *be the real part of a harmonizable process, i.e.,*

$$\{X_t\}_{t>0} \stackrel{d}{=} \{ \int_{[0,2\pi)\times \mathbf{R}} t^H \cos(s + w \log t)\, Z(ds, dw)\}_{t>0},$$

where Z *is a real-valued* $S\alpha S$ *random measure with control measure* $Leb \otimes \nu$ *and* ν *is a finite measure on* $\mathbf{R}$ *(see [Ros 2], Example 4.9). Here* $\phi_t(s, w) = (s +_{2\pi} w \log t, w)$, *where "* $+_{2\pi}$ *" denotes addition modulo* 2π.

Theorem 4.4 *Every* $S\alpha S$ *self-similar process* $\{X_t\}_{t>0}$ *admits a unique decomposition into three independent parts*

$$\{X_t\}_{t>0} \stackrel{d}{=} \{X_t^{(1)}\}_{t>0} + \{X_t^{(2)}\}_{t>0} + \{X_t^{(3)}\}_{t>0},$$

where the first process on the right-hand side is a MFM, the second is harmonizable, and the thrid one is an $H - ss$ evanescent process.

Proof. Since the set D of Hopf decomposition and the set of fixed points for a flow are invariant, we obtain a decomposition of self-similar processes analogous to the decomposition of stationary processes (see Theorem 6.1 in [Ros 1]). □

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