# STABLE CARMA PROCESSES

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# Introduction

We investigate the new class of stochastic processes that are used in the financial mathematics, i.e. CARMA processes (continuous ARMA models) with symmetric stable innovations, that are a natural extension of second-order Lévy-driven CARMA processes. They are also the extension of ARMA models with symmetric $\alpha$-stable innovations. For the considered stable models the covariance function is not defined and therefore other measures of dependence have to be used. We present the form of solution of considered continuous models and study the codifference and the covariation - the most popular measures of dependence defined for symmetric $\alpha$-stable processes. We show the codifference and the covariation are asymptotically proportional with the coefficient of proportionality equal to $\alpha$.

# CARMA processes with symmetric $\alpha$-stable Lévy motion

The CARMA(p,q) process with symmetric $\alpha$-stable Lévy motion indexed by $\mathbb{R}$, it is the processes satisfying following equation

$$a(D)Y(t) = b(D)DL^*(t), \; t \in \mathbb{R}, \tag{1}$$

in which $D$ denotes differentiation with respect to $t$,

$$a(z) = z^p + a_1 z^{p-1} + \cdots + a_p,$$
$$b(z) = b_0 + b_1 z + \ldots b_{p-1} z^{p-1},$$

and the coefficients $b_j$ satisfy $b_q \neq 0$ and $b_j = 0$ for $q < j < p$. For simplicity we study the special case of such models, i.e. the symmetric $\alpha$-stable CARMA(1,1) processes:

$$DY(t) + aY(t) = bDL^*(t), \tag{2}$$

for non-zero $a$ and $b$ parameters. In equations (1) and (2) $\{L^*(t),\ t \in \mathbb{R}\}$ is an $\alpha-$stable Lévy process defined as follows:

$$L^*(t) = V(t)I_{[0,\infty)}(t) - Z(-t)I_{[-\infty,0)}(t),\ -\infty < t < \infty. \tag{3}$$

where $\{V(t),\ t \geq 0\}$ and $\{Z(t),\ t \geq 0\}$ are two independent symmetric $\alpha-$stable Lévy motions with the same $\alpha$ parameter. We will make the assumption that $1 < \alpha \leq 2$.

# Form of the solution

The process $\{Y(t),\ t \in \mathbb{R}\}$ defined as

$$Y(t) = b \int_{-\infty}^{t} e^{-a(t-u)} dL^*(u), \tag{4}$$

satisfies equation (2).

- The process $\{Y(t),\ t \in \mathbb{R}\}$ given in (4) for $a > 0$ and $b = 1$ is called an $\alpha-$stable Ornstein-Uhlenbeck process.
- For $a > 0$ the stochastic process $\{Y(t),\ t \in R\}$ given in (4) is stationary.

# Measures of dependence for stable random variable

Let $X$ and $Y$ be jointly symmetric $\alpha-$stable random variables (S$\alpha$S for short) and let $\Gamma$ be the spectral measure of the random vector $(X, Y)$. The most popular measures of dependence for stable random variables are the covariation $CV(X, Y)$ and the codifference $CD(X, Y)$ (see [6]).

**Definition 1** *Let $X$ and $Y$ be jointly S$\alpha$S. The covariation $CV(X,Y)$ of $X$ on $Y$ defined for $1 < \alpha \leq 2$ is the real number*

$$CV(X,Y) = \int_{S_2} s_1 s_2^{<\alpha-1>} \Gamma(d\mathbf{s}), \tag{5}$$

*where $\Gamma$ is the spectral measure of the random vector $(X, Y)$, $\mathbf{s} = (s_1, s_2)$ and the signed power $z^{<p>}$ is given by $z^{<p>} = |z|^{p-1}\bar{z}$.*

**Definition 2** *Let $X$ and $Y$ be jointly S$\alpha$S. The codifference $CD(X,Y)$ of $X$ on $Y$ defined for $0 < \alpha \leq 2$ equals*

$$CD(X,Y) = \ln \mathbb{E} \exp\{i(X-Y)\} - \ln \mathbb{E} \exp\{iX\} - \ln \mathbb{E} \exp\{-iY\}. \quad (6)$$

In contrast to the codifference, the covariation is not symmetric in its arguments. Moreover, when $\alpha = 2$ both measures reduce to the covariance, namely

$$Cov(X,Y) = 2CV(X,Y) = CD(X,Y). \quad (7)$$

If $\alpha > 1$, then the covariation induces a norm $||.||_\alpha$ on the linear space $S_\alpha$ of jointly S$\alpha$S random variables.

**Definition 3** *The covariation norm of* $X \in S_\alpha$, $\alpha > 1$, *is*

$$||X||_\alpha = (CV(X,X))^{1/\alpha}. \tag{8}$$

The covariation norm of a S$\alpha$S random variable $X$ is equal to the scale parameter of this variable. For $1 < \alpha \leq 2$ the codifference of S$\alpha$S random variables X and Y can be rewritten in the form

$$CD(X,Y) = ||X||_\alpha^\alpha + ||Y||_\alpha^\alpha - ||X-Y||_\alpha^\alpha. \tag{9}$$

# CARMA(1,1) processes with symmetric $\alpha$-stable Lévy motion. Measures of dependence.

**Proposition 1** *Let* $\{Y(t),\ t \in \mathbb{R}\}$ *be the solution of (2) and* $a > 0$, *then for* $1 < \alpha \leq 2$ *the covariation of* $Y(t)$ *on* $Y(s)$ *for* $s, t \in \mathbb{R}$ *has the following form:*

$$CV(Y(t), Y(s)) = \begin{cases} |b|^{\alpha} \dfrac{e^{-a(t-s)}}{a\alpha} & \text{for } s < t, \\ |b|^{\alpha} \dfrac{e^{a(\alpha-1)(t-s)}}{a\alpha}, & \text{for } s \geq t. \end{cases}$$

**Proposition 2** *If* $\{Y(t),\ t \in \mathbb{R}\}$ *is the solution of (2) and* $a > 0$, *then for* $1 < \alpha \le 2$ *the codifference of* $Y(t)$ *on* $Y(s)$ $(s, t \in \mathbb{R})$ *has the following form:*

$$CD(Y(t), Y(s)) = |b|^{\alpha} \frac{1 + e^{-a\alpha|t-s|} - |1 - e^{-a|t-s|}|^{\alpha}}{a\alpha}.$$

**Theorem 1** *If* $\{Y(t),\ t \in \mathbb{R}\}$ *is the solution of equation (2) and* $a > 0$, *then:*

*(a) for* $1 < \alpha \leq 2$ *and each* $t \in \mathbb{R}$ *the following formula holds:*

$$\lim_{h\to\infty} \frac{CD(Y(t), Y(t-h))}{CV(Y(t), Y(t-h))} = \lim_{h\to\infty} \frac{CD(Y(t+h), Y(t))}{CV(Y(t+h), Y(t))} = \alpha, \quad (10)$$

*(b) for* $1 < \alpha < 2$ *and each* $t \in \mathbb{R}$ *the following formula holds:*

$$\lim_{h\to\infty} \frac{CD(Y(t-h), Y(t))}{CV(Y(t-h), Y(t))} = \lim_{h\to\infty} \frac{CD(Y(t), Y(t+h))}{CV(Y(t), Y(t+h))} = 0. \quad (11)$$

# Examples

**Example 1 The discrete version of (2).**
*Let us consider the discrete version of considered SαS CARMA(1,1) processes defined in (2), i.e. time series model given by the equation:*

$$X(n) + aX(n-1) = b\xi(n), \tag{12}$$

*where $a, b \neq 0$ are real numbers and $\{\xi(n), n \in Z\}$ are independent, SαS innovations with the scale parameter $\sigma_\xi$. The time series $\{X(n),\ n \in Z\}$ satisfying (12) is the special case of presented in Nowicka [3] ARMA models with SαS innovations. This is also the special case of PARMA models with SαS innovations considered in Nowicka-Zagrajek and Wyłomańska [4] and described in Nowicka-Zagrajek and Wyłomańska [5] ARMA models with time-varying coefficients and SαS innovations. Using results obtained in Nowicka-Zagrajek and Wyłomańska [4] and Nowicka-Zagrajek and*

*Wyłomańska [5] we obtain the following formulas for the coddiference and covariation under the assumption $|a| < 1$:*

$$CV(X(n), X(n-k)) = CV(X(n+k), X(n)) = \frac{(-a)^k |\sigma_\xi b|^\alpha}{1-|a|^\alpha}, \quad (13)$$

$$CV(X(n), X(n+k)) = CV(X(n-k), X(n)) = \frac{|a|^{\alpha k} |\sigma_\xi b|^\alpha}{(-a)^k (1-|a|^\alpha)}, \quad (14)$$

$$\begin{aligned} CD(X(n), X(n-k)) &= CD(X(n+k), X(n)) \\ &= \frac{\sigma_\xi^\alpha (1 + |a|^{\alpha k} - |1-(-a)^k|^\alpha)|b|^\alpha}{1-|a|^\alpha}, \quad (15) \end{aligned}$$

*for every $n \in Z$ and for $k \in Z, k > 0$. Results 13 – 15 are not surprising because of the stationarity of $X(n)$.*

*It is not difficult to show that for $1 < \alpha < 2$ and for every $n \in Z$ we obtain the similar results as in the continuous case:*

*(a)*

$$\lim_{k\to\infty} \frac{CD(X(n+k), X(n))}{CV(X(n+k), X(n))} = \lim_{k\to\infty} \frac{CD(X(n), X(n-k))}{CV(X(n), X(n-k))} = \alpha,$$

*(b)*

$$\lim_{k\to\infty} \frac{CD(X(n-k), X(n))}{CV(X(n-k), X(n))} = \lim_{k\to\infty} \frac{CD(X(n), X(n+k))}{CV(X(n), X(n+k))} = 0.$$

**Example 2** *In order to illustrate our theoretical results let us consider SαS CARMA(1,1) process:*

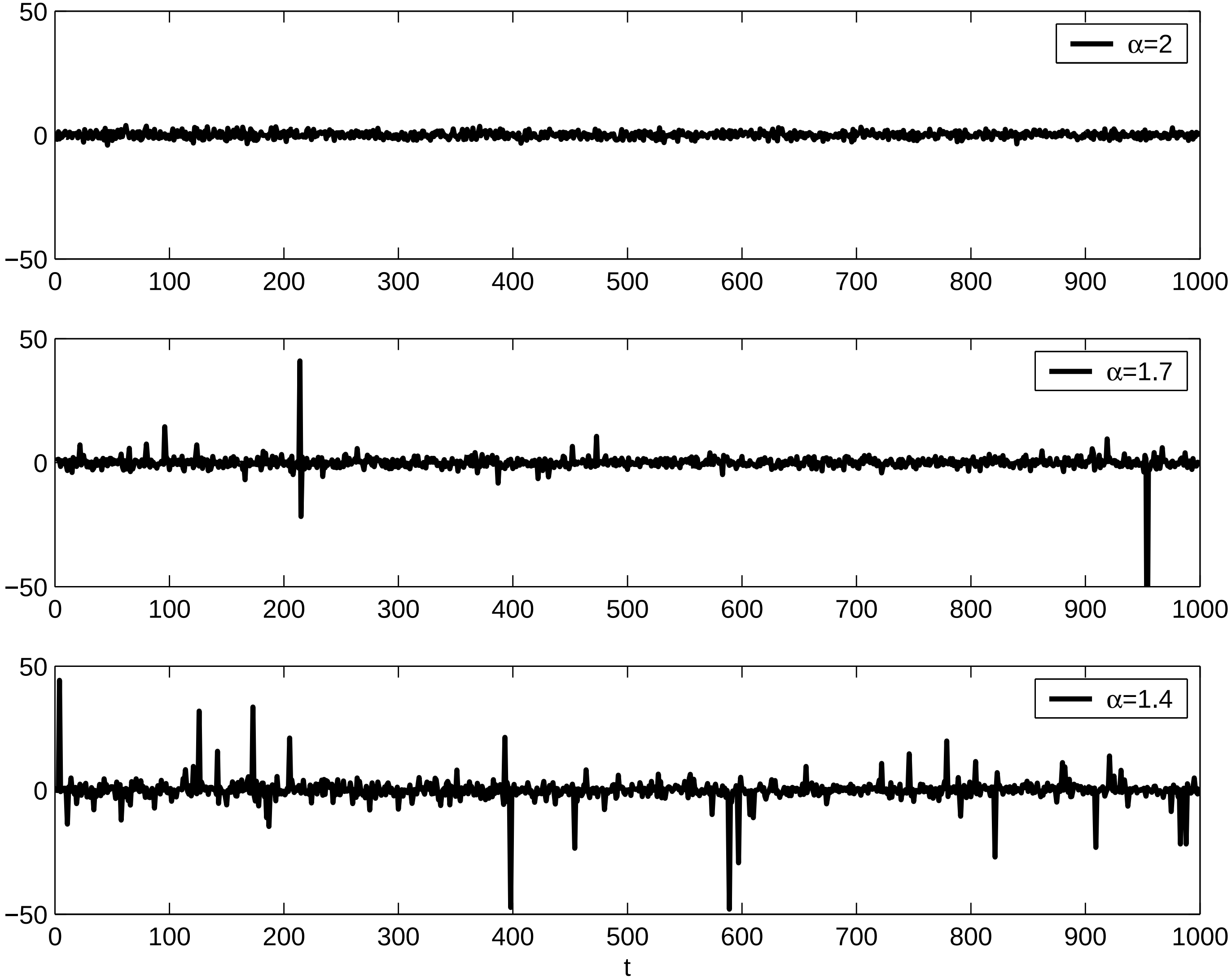
$$DY(t) + 0.5Y(t) = DL^*(t),$$

*for* $\{L^*(t),\ t \in \mathbb{R}\}$ *is given in (3) and* $1 < \alpha \leq 2$*.*
*As a comparison let us take into account the corresponding to the process* $\{Y(t),\ t \in \mathbb{R}\}$ *the discrete ARMA(1,1) model with the same parameters given by:*

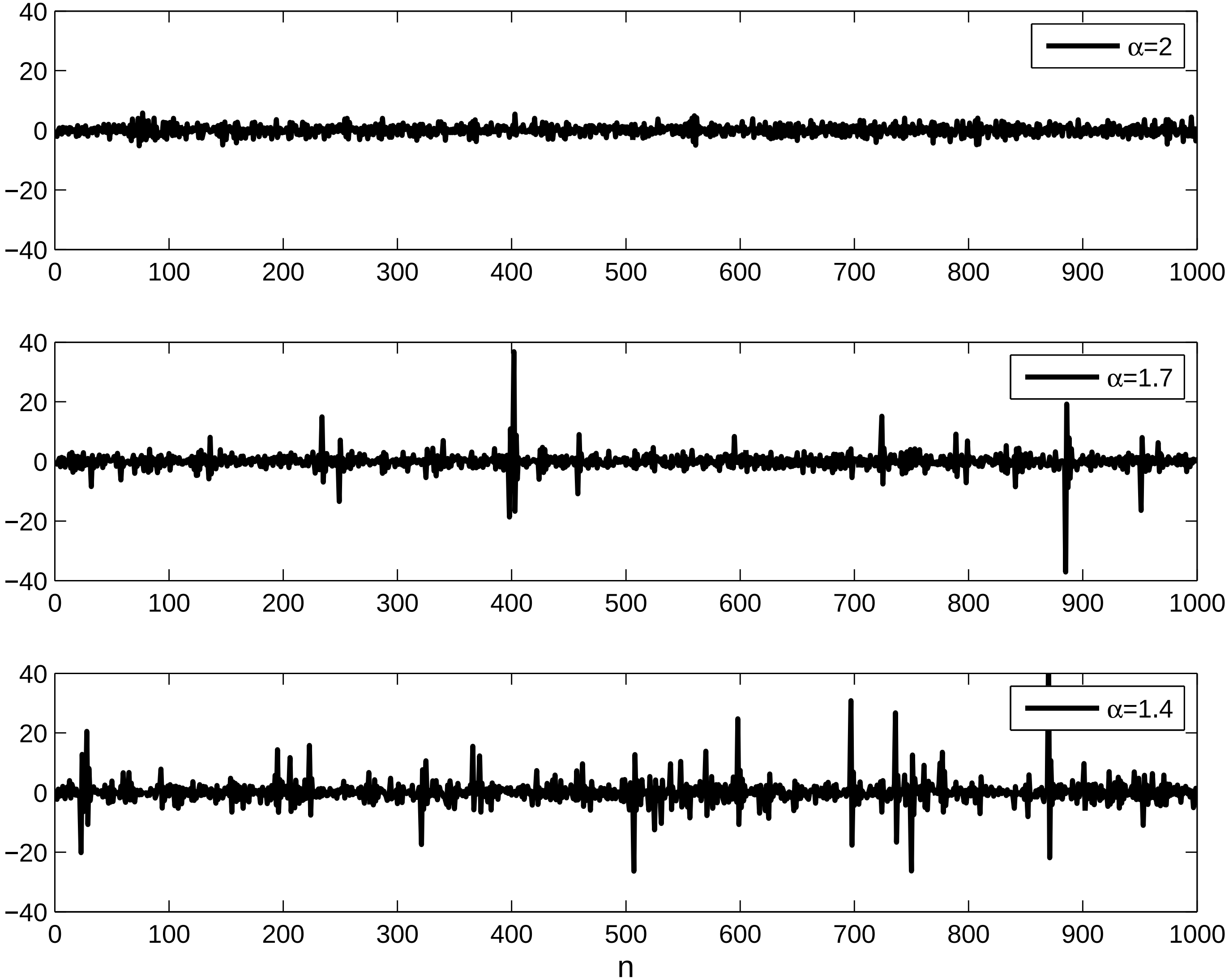
$$X(n) + 0.5X(n-1) = \xi(n),$$

*where* $\{\xi(n),\ n \in \mathbb{R}\}$ *are independent SαS random variables with* $1 < \alpha \leq 2$*.*

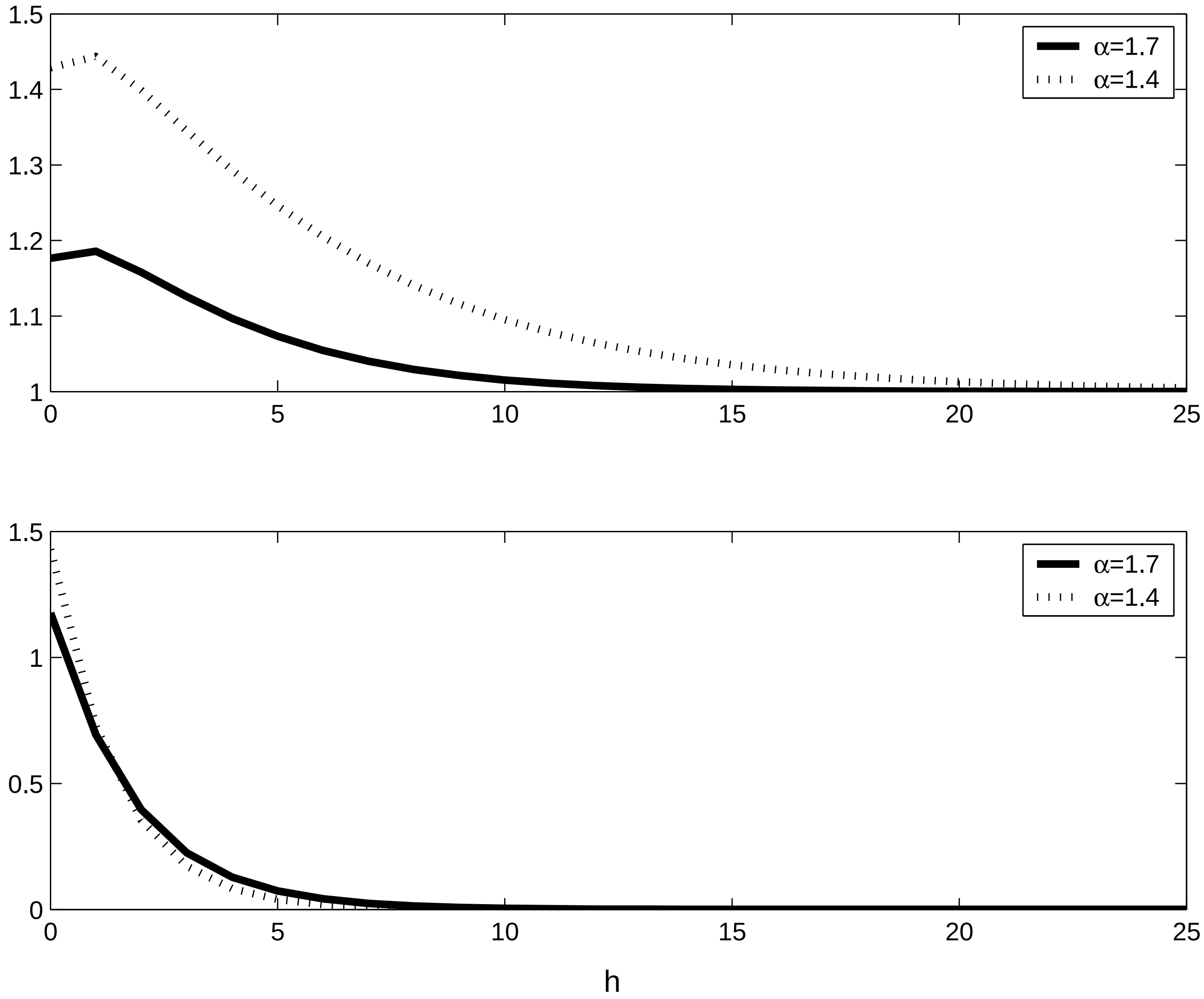
## *Realizations of SαS CARMA(1,1) process*



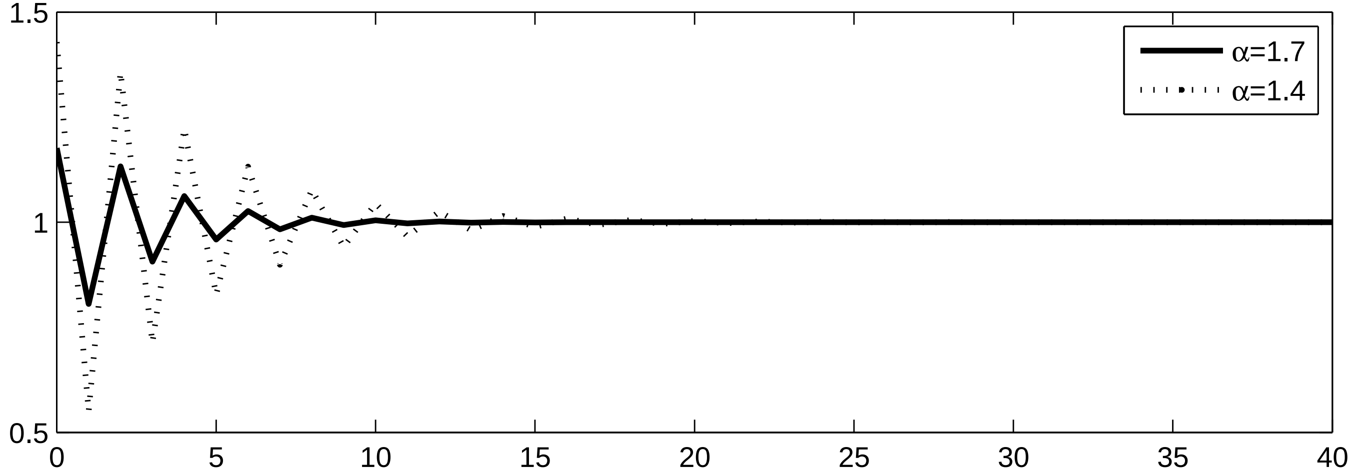
*Realizations of ARMA(1,1) model with SαS innovations*

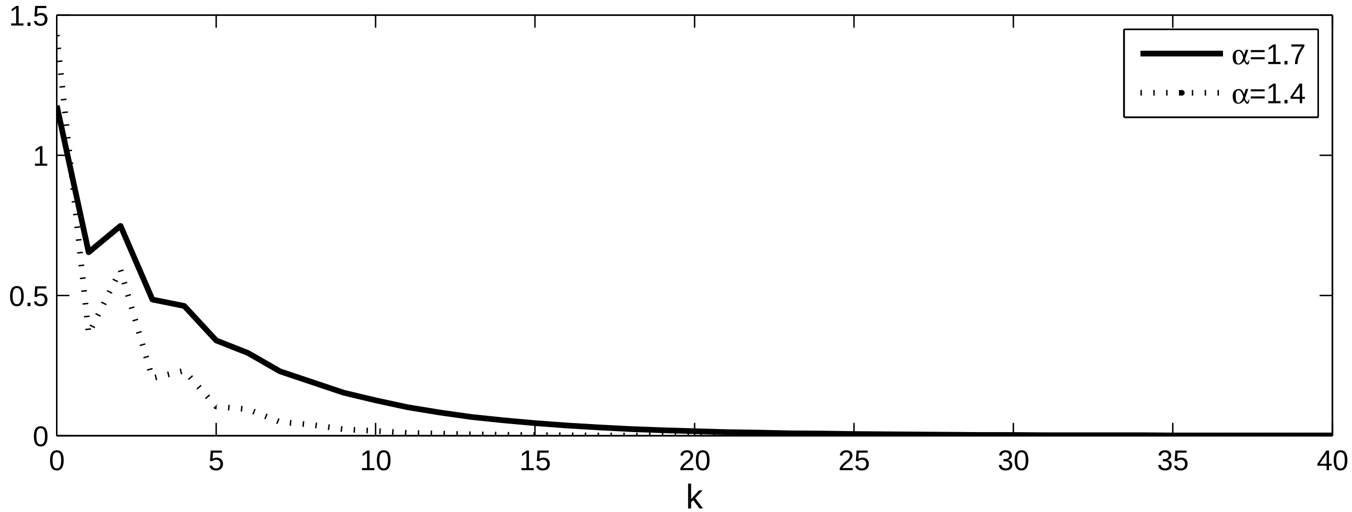


## *Functions* $\frac{CD(Y(t+h),Y(t))}{\alpha CV(Y(t+h),Y(t))}$ *and* $\frac{CD(Y(t),Y(t+h))}{\alpha CV(Y(t),Y(t+h))}$



*Functions* $\frac{CD(X(n+k),X(n))}{\alpha CV(X(n+k),X(n))}$ *and* $\frac{CD(X(n),X(n+k))}{\alpha CV(X(n),X(n+k))}$



# Applications

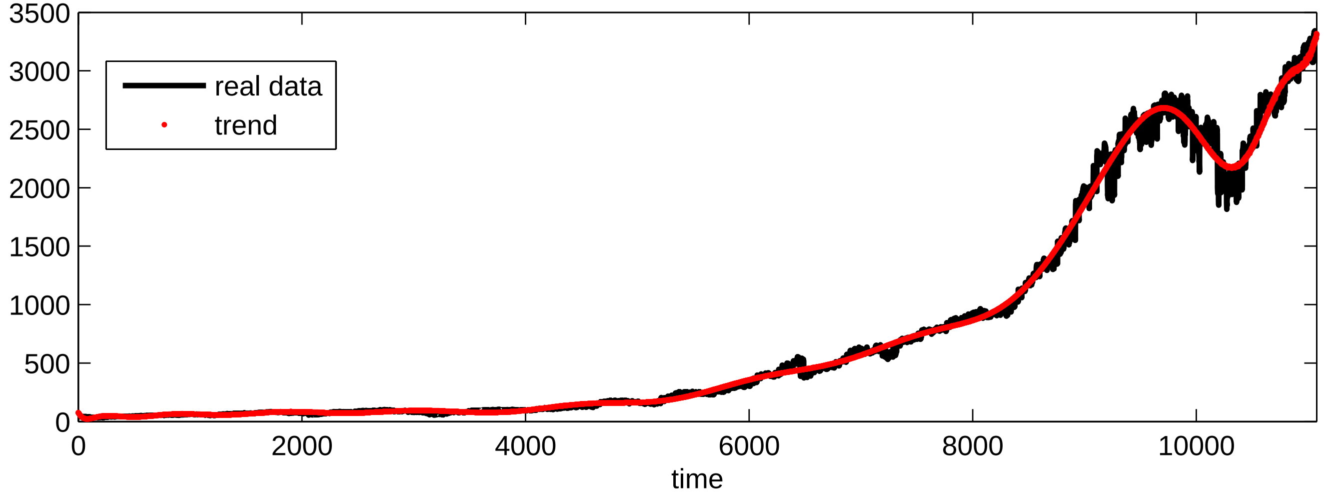
As an application to stochastic volatility modelling, Berndorff-Nielsen and Shephard ([1]) introduced a model for asset-pricing in which the logarithm of an asset price is the solution of the stochastic differential equation:

$$DY(t) = \mu + \beta\sigma^2(t) + \sigma(t)DW(t),$$

where $\{\sigma^2(t)\}$, the instantaneous volatility, is a non-negative Lévy-driven Ornstein-Uhlenbeck process, $\{W(t)\}$ is standard Brownian-motion and $\mu$ and $\beta$ are constants (see [2]).

Much of the analysis of Berndorff-Nielsen and Shephard can however be carried out after replacing the Ornstein-Uhlenbeck process by a symmetric a-stable CARMA process with $\alpha = 2$ (second order CARMA process). This has the advantage of allowing the representation of volatility processes with a larger range of autocorrelations functions than is possible in the Ornstein-Uhlenbeck framework. Brockwell and Marquardt in [2] propose the take CARMA(3,2).

Let us consider the index level associated with the return on the indexes for NYSE and AMEX for the period 1962-2005. On the next Figure we present the considered time series and the trend that was removed before the further analysis.


3500
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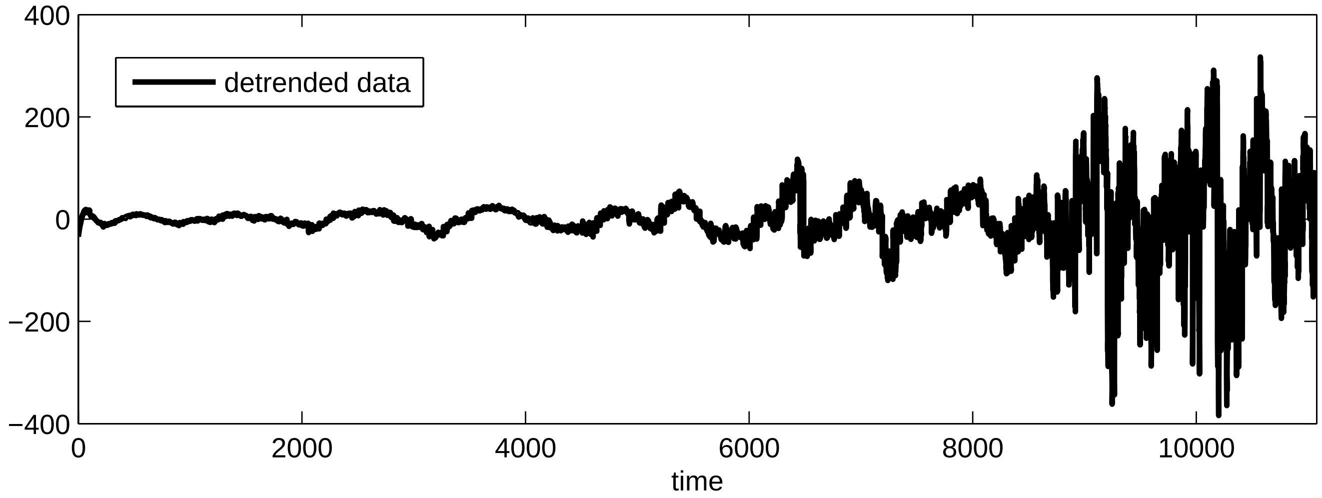
6000
8000
10000
time
real data
trend


400
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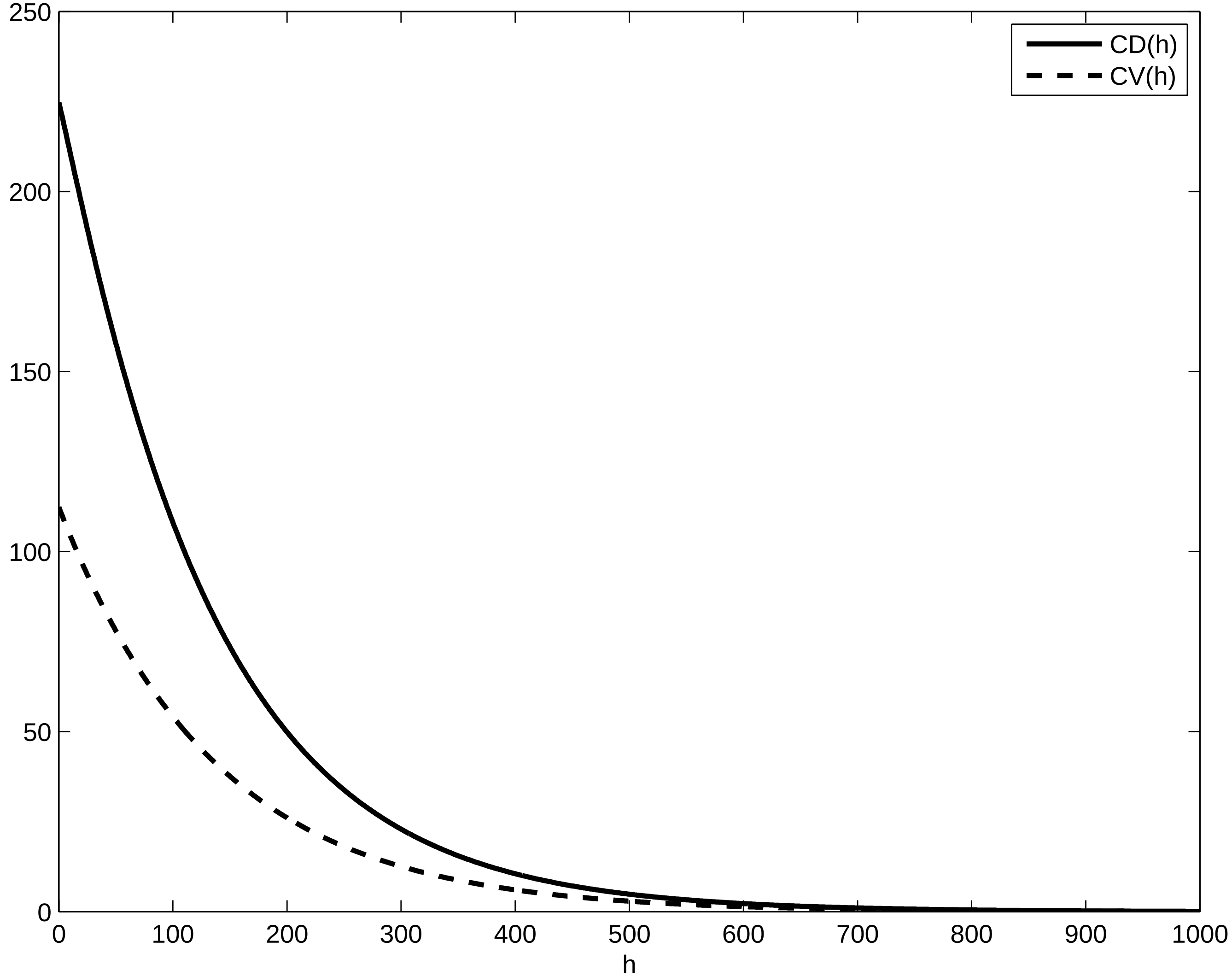
−400
0
2000
4000
6000
8000
10000
time
detrended data

The data can be modelled by the CARMA process. As a best model we choose the CAR(1) given by the formula

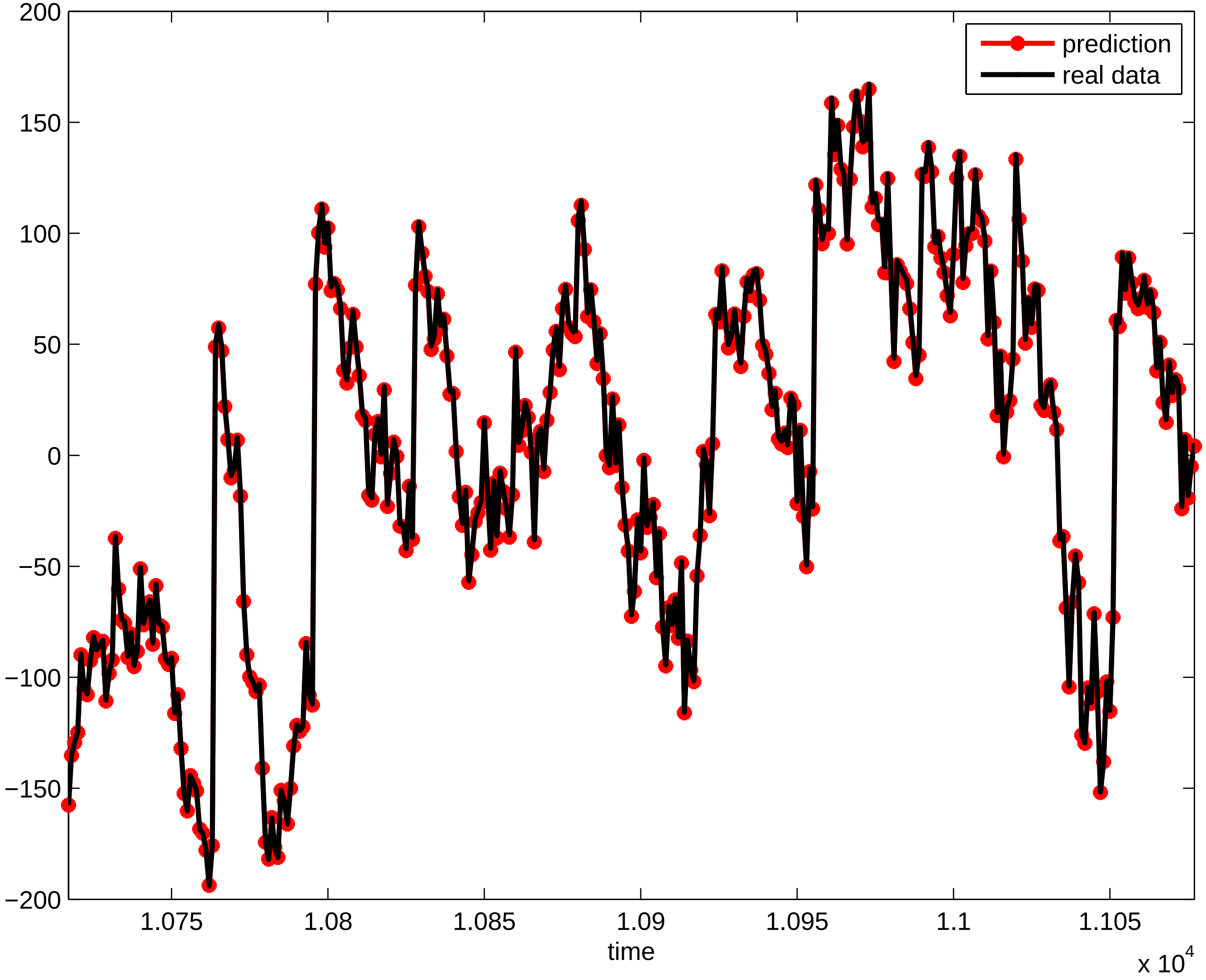
$$DY(t) + 0.0073Y(t) = DL^*(t),$$

where $\{L^*(t),\ t \in \mathbb{R}\}$ is a two-sided symmetric $\alpha-$stable Lévy motion with $\alpha = 1.219$. The Jarque-Bera test as well as the Lilliefors test reject the hypothesis that the residuals come from a distribution in the normal family.

On the next Figure we illustrate the asymptotic behaviour of the coddiference $(CD(h))$ and covariation $(CV(h))$ for $h \in [0, 1000]$ obtained as an estimator based on the estimated CAR(1) parameter and index of stability.


250
200
150
100
50
0
0
100
200
300
400
500
600
700
800
900
1000
h
CD(h)
CV(h)

On the next Figure we present the one-step prediction for the year 2005. The data from that year were not taken to the previous estimation. The mean prediction error is equal 3.6163%.


200
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100
50
0
−50
−100
−150
−200
1.075
1.08
1.085
1.09
1.095
1.1
1.105
time
x 10^4
prediction
real data

# References

[1] O. Barndorff-Nielsen and N. Shepardt, Non-gaussian Ornstein-Uhlenbeck-based models and some of their uses in financial economics. *J. Roy.Statist. Soc.Ser. B* **63** 1-42 (2001).

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