

SUBDYNAMICS OF FINANCIAL DATA FROM  
FRACTIONAL FOKKER–PLANCK EQUATION\*

JOANNA JANCZURA, AGNIESZKA WYŁOMAŃSKA

Institute of Mathematics and Computer Science  
Hugo Steinhaus Center, Wrocław University of Technology  
Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland*(Received April 1, 2009)*

In exhibition of many real market data we observe characteristic traps. This behavior is especially noticeable for processes corresponding to stock prices. Till now, such economic systems were analyzed in the following manner: before the further investigation trap-data were removed or omitted and then the conventional methods used. Unfortunately, for many observations this approach seems not to be reasonable therefore, we propose an alternative attitude based on the subdiffusion models that demonstrate such characteristic behavior and their corresponding probability distribution function (pdf) is described by the fractional Fokker–Planck equation. In this paper we model market data using subdiffusion with a constant force. We demonstrate properties of the considered systems and propose estimation methods.

PACS numbers: 05.10.–a, 02.50.Ey, 02.70.–c, 89.65.Gh

**1. Introduction**

Subdiffusion described by the fractional Fokker–Planck equation (FFPE) plays an important role in statistical physics. It has found an application in many areas like polymeric networks, porous systems, nuclear magnetic resonance and transport on fractal objects [1–3]. The most recognizable features of the subdiffusive dynamics are traps — periods when the test particle stays motionless. In the general definition of the FFPE the fractional derivative of the Riemann–Liouville type [4]  ${}_0D_t^{1-\alpha}$  defined in (3) is responsible for the power-law behavior of the mean square displacement, namely  $\langle Y^2(t) \rangle \propto t^\alpha$ ,  $0 < \alpha < 1$ , as well as for the heavy-tailed waiting times in the corresponding CTRW scheme. Thus, the fractional derivative distinguishes the FFPE from the traditional Fokker–Planck equation, related to the classical diffusion.

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\* Presented at the XXI Marian Smoluchowski Symposium on Statistical Physics Zakopane, Poland, September 13–18, 2008.

In economics the most classical and still popular approach is the Black–Scholes model based on the Brownian diffusion process. Although there are many generalizations of this model (for instance [5–7]), most of them do not capture important feature observed in market data, namely periods of constant values. The idea of subdiffusion deals with this problem.

We present two examples of economic data exhibiting subdiffusive behavior. The first one comes from the Polish Stock Exchange, while the second describes prices from the Nordic Exchange Baltic Market. Regarding characteristics of considered data we choose the suitable fractional Fokker–Planck equation with a constant force. The subordinated process defined in (1) is the stochastic representation of the FFPE, [8, 9] (similar relation in case of space-dependent force is shown in [10]). The approach based on the link between Langevin-type dynamics and the fractional Fokker–Planck equation allows us to analyze many characteristics of such processes, such as moments and quantile lines by using the Monte Carlo methods [10–13]. In the estimation procedure leading to an appropriate subdiffusion model adequate to real data the most important issue is related to the estimation of  $\alpha$  parameter that is connected with fractional part in FFPE corresponding to the analyzed process. On the other hand  $\alpha$  is responsible for observed traps in subordinated process. Therefore, the estimation procedure should be based on trap-data. In this paper we attempt to use such approach to estimate unknown parameter  $\alpha$ .

## 2. Subdiffusion process with constant drift

The subordinated process is defined as follows [8]:

$$Y(t) = X(S_\alpha(t)), \quad (1)$$

where  $\{S_\alpha(t)\}_{t \geq 0}$  is inverse  $\alpha$ -stable subordinator of  $\{U_\alpha(\tau)\}_{\tau \geq 0}$  [14, 15], *i.e.*

$$S_\alpha(t) = \inf\{\tau > 0 : U_\alpha(\tau) > t\} \quad (2)$$

for  $\alpha$ -stable nondecreasing Levy process  $\{U_\alpha(\tau)\}_{\tau \geq 0}$  [16] with the Laplace transform  $E(e^{-uU_\alpha(\tau)}) = e^{-\tau u^\alpha}$ ,  $0 < \alpha < 1$  and  $\{X(\tau)\}_{\tau \geq 0}$  satisfies the following stochastic differential equation with respect to the Brownian motion  $\{B(\tau)\}_{\tau \geq 0}$ :

$$dX(\tau) = F d\tau + dB(\tau), \quad X(0) = 0,$$

with constant force  $F$ . Moreover,  $B(\tau)$  and  $S_\alpha(t)$  are assumed to be independent. The pdf of the process  $\{Y(t)\}_{t \geq 0}$  is given in [9] by the fractional Fokker–Planck equation [8]:

$$\frac{\partial w(x, t)}{\partial t} = \left[ -F \frac{\partial}{\partial x} + \frac{1}{2} \frac{\partial^2}{\partial x^2} \right]_0 D_t^{1-\alpha} w(x, t), \quad w(x, 0) = \delta(x),$$

where the fractional derivative of the Riemann–Liouville type [4] is defined as follows for  $f \in C^1([0, \infty))$  and  $\alpha \in (0, 1)$ :

$${}_0D_t^{1-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \frac{d}{dt} \int_0^t (t-s)^{\alpha-1} f(s) ds. \tag{3}$$

Using the form of the solution of the process  $\{Y(t)\}_{t \geq 0}$  [8] and methods of calculating integrals of inverse subordinators [17] we obtain

$$\langle Y(t) \rangle = \frac{F}{\Gamma(\alpha + 1)} t^\alpha, \quad \langle Y^2(t) \rangle = \frac{2F^2}{\Gamma(2\alpha)} t^{2\alpha} + \frac{1}{\Gamma(\alpha + 1)} t^\alpha. \tag{4}$$

The same result might be shown using the recursive relation between the moments proved in [9].

The explicit formulas for measures of dependence (covariance and correlation functions) of the considered process with  $F = 0$  can be calculated by using the fact that  $\{Y(t)\}_{t \geq 0}$  in that case is a martingale with respect to the  $\sigma$ -field  $G_t$  defined in details in [18]. Namely, the theoretical covariance function  $\text{cov}(t, s) = \langle Y(t), Y(s) \rangle - \langle Y(t) \rangle \langle Y(s) \rangle$  is as follows:

$$\text{cov}(t, s) = \begin{cases} \frac{s^\alpha}{\Gamma(\alpha+1)} & \text{for } s \leq t \\ \frac{t^\alpha}{\Gamma(\alpha+1)} & \text{for } s > t. \end{cases} \tag{5}$$

As we observe the covariance depends only on  $\min\{t, s\}$  what indicates that increments of the analyzed process are non-stationary. The similar problem is also considered in [19]. In that simple case the correlation function is given by:

$$\text{corr}(t, s) = \frac{\text{cov}(t, s)}{\sqrt{\langle Y(t)^2 \rangle \langle Y(s)^2 \rangle}} = \left( \frac{\min\{t, s\}}{\max\{t, s\}} \right)^{\alpha/2}.$$

The form of the subordinated process (1) describing subdiffusion in terms of stochastic processes provides a powerful analysis tool, namely simulations and Monte Carlo methods. For a detailed description of the simulation procedure, see [20].

In Fig. 1 we present mean and variance of the considered subdiffusion process obtained from Monte Carlo simulations with 10000 iterations. Moreover, in Fig. 2 we demonstrate the quantile lines and two sample trajectories of the process  $\{Y(t)\}_{t \geq 0}$ . The covariance based on the 1000 trajectories of subordinated process with  $F = 0$ ,  $\alpha = 0.7$  and theoretical function  $\text{cov}(t, s)$  given in formula (5) for fixed value of  $s = 5$  is shown in Fig. 3. As we observe the presented functions are constant for  $s > 5$ .

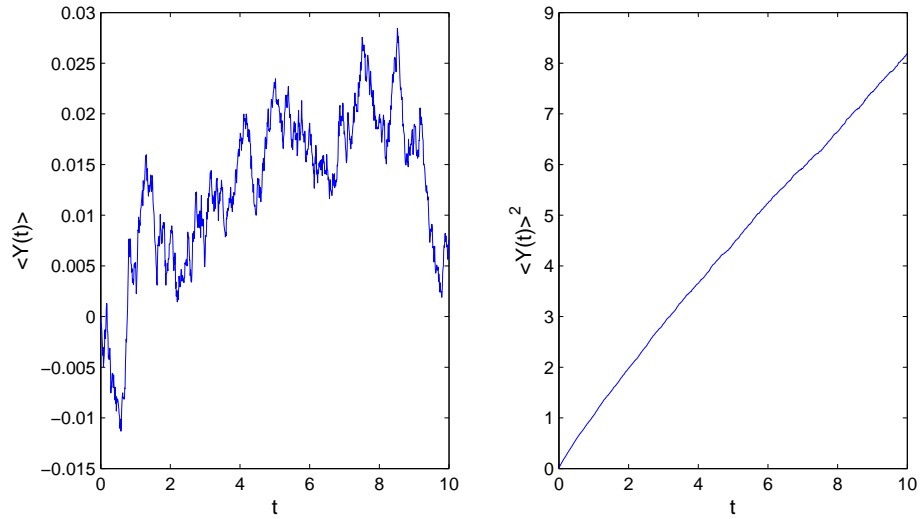


Fig. 1. Mean (left panel) and variance (right panel) of the subdiffusion process with  $F = 0$  and  $\alpha = 0.9$  obtained from Monte Carlo simulations with 10000 iterations. The values are consistent with explicit formulas (4).

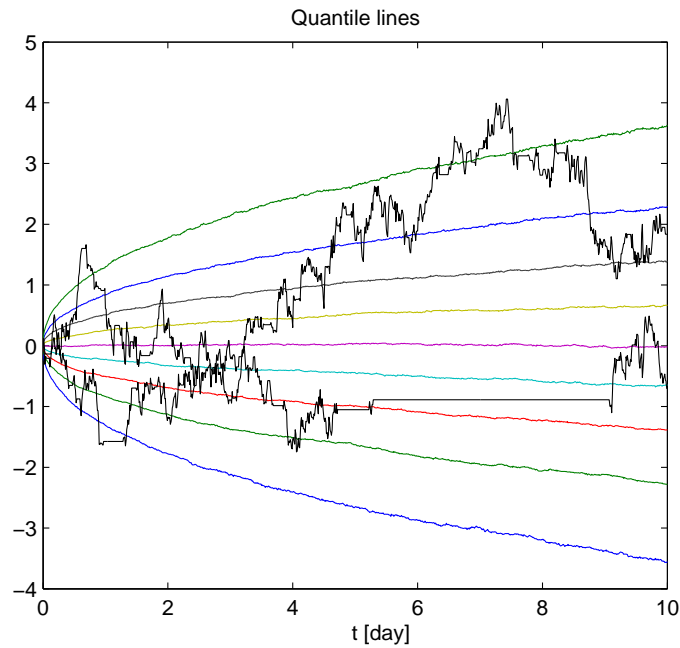


Fig. 2. Quantile lines and two sample trajectories of the subdiffusion process with  $F = 0$  and  $\alpha = 0.9$  obtained from Monte Carlo simulations with 10000 iterations.

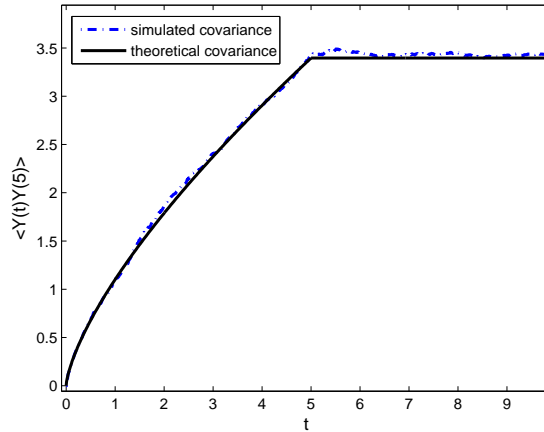


Fig. 3. The covariance based on the 1000 trajectories of the subordinated process with  $F = 0$ ,  $\alpha = 0.7$  and corresponding the theoretical function  $\text{cov}(t, s)$  given in formula (5) for fixed value of  $s = 5$ .

### 3. Estimation of the index $\alpha$

In order to apply subdiffusion model (1) to real market data it is crucial to give parameters estimation procedures. Here, we focus on the  $\alpha$ -parameter estimation.

As we know, the specific traps in subdiffusion trajectories occur when the time process  $S_\alpha(t)$  (2) is constant for some period. As  $S_\alpha(t)$  is the inverse  $\alpha$ -stable subordinator, the length of constant periods has a totally skewed  $\alpha$ -stable distribution. In Fig. 4 we illustrate relation between process  $S_\alpha(t)$  and  $U_\alpha(t)$ . In order to find parameter  $\alpha$  we calculate the sizes of traps and treat them as independent and identically distributed (i.i.d.)  $\alpha$ -stable random variables. Parameter  $\alpha$  is then estimated by using methods known for  $\alpha$ -stable distributions.

In our study we consider six methods of  $\alpha$ -estimation. The first one, Hill estimation method [21] is based on the assumption that the upper tail of the distribution is of the form  $1 - F(x) \sim x^{-\alpha}$  (heavy tail behavior). If  $X_{(1)}, X_{(2)}, \dots, X_{(N)}$  are the order statistics of the analyzed sample from the population with cumulative distribution function (cdf)  $F$  such that  $X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(N)}$ , then the Hill estimate based on the  $k$  largest statistics is given by the following equation:

$$\alpha_{\text{Hill}}(k) = \left( \frac{1}{k} \sum_{n=1}^k \log \frac{X_{(N-n+1)}}{X_{(N-k)}} \right)^{-1}.$$

For the more detailed description of this method and its main features see [22].

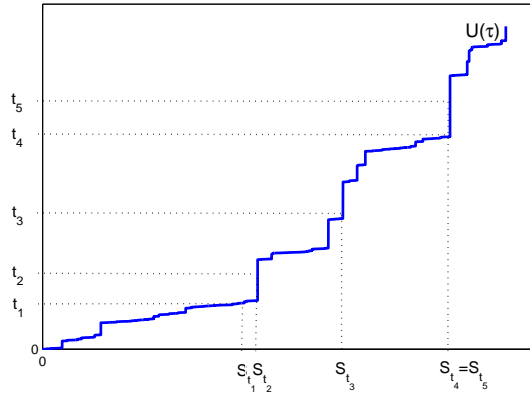


Fig. 4. The relation between the process  $U_\alpha(\tau)$  and its inverse subordinator  $S_\alpha(t)$ . Observe that constant periods of  $S_\alpha(t)$  occur accordingly to the jumps of the process  $U_\alpha(\tau)$ .

The second considered method proposed in [23], the Pickands estimate, similar as the Hill method, is based on the heavy-tail behavior. The formula for the estimate that considers  $k$  largest order statistics is as follows:

$$\alpha_{\text{Pickands}}(k) = \log 2 \left( \log \left( \frac{X_{(N-\lfloor k/4 \rfloor)} - X_{(N-\lfloor k/2 \rfloor)}}{X_{(N-\lfloor k/2 \rfloor)} - X_{(N-k)}} \right) \right)^{-1}.$$

The third method of  $\alpha$  estimation, the type of EVI estimate (Extreme Value Index) [24] based on the  $k$  largest order statistics has the following form:

$$\alpha_{\text{EVI}}(k) = \left( \frac{1}{\alpha_{\text{Hill}}(k)} + 1 - \frac{1}{2} \left( 1 - \frac{H^{(2^*)}(k)}{\alpha_{\text{Hill}}(k)^2} \right)^{-1} \right)^{-1},$$

where

$$H^{(2^*)}(k) = \left( \frac{1}{k} \sum_{n=1}^k \left( \log \frac{X_{(N-n+1)}}{X_{(N-k)}} \right)^2 \right)^{-1}.$$

Unfortunately, for the three proposed methods it is difficult to choose the right value of  $k$ . In practice, the estimates are plotted against  $k$  and one looks for the region where the plot levels off to identify the correct order statistics [22]. Therefore, in our study for simulated data as well as for real financial data as an estimate we give the range of obtained statistics.

The POT estimate (Peaks Over Threshold) of  $\alpha$  parameter [23, 25] is based on the Pickands–Balkema–de Haan theorem that the conditional cdf

$$F_u(x) = P(X - u \leq x | X > u), \quad u \geq 0, x > 0,$$

where  $X$  has  $\alpha$ -stable distribution, can be approximated by cdf of Generalized Pareto Distribution (GPD) having the following form:

$$G_\alpha(x) = 1 - \left(1 + \frac{x}{\alpha}\right)^{-\alpha}.$$

In this method for a given sample  $X_1, X_2, \dots, X_N$  we select a high threshold  $u$  and denote  $N_u$  as the number of exceedances above  $u$ . Moreover, we denote  $Y_k = X_{j_k} - u$  for  $k = 1, 2, \dots, N_u$  such that  $X_{j_k} > u$ . Next, we fit GPD to the excesses  $Y_1, Y_2, \dots, Y_{N_u}$  by using maximum likelihood method to estimate the parameter  $\alpha$ .

The next estimation method, the M–S estimate, is proposed by Meerschaert and Scheffler in [26]. It is defined for  $N$  observations  $X_1, X_2, \dots, X_N$  as follows

$$\alpha_{\text{M-S}} = \frac{2(\gamma + \log(N))}{\gamma + \log_+ \sum_{i=1}^N (X_i - \langle X \rangle)^2},$$

where  $\gamma = 0.5772$  is the Euler’s constant,  $\langle X \rangle$  — the sample mean and  $\log_+(x) = \max(0, \log(x))$ . For data from heavy tail distribution these asymptotics dependent on the tail index and not on the exact form of the distribution [26].

In the last considered estimation method, PCF (Power Curve Fitting), we assume that the given sample  $X_1, X_2, \dots, X_N$  is generated as a sequence of i.i.d. random variables with heavy tail behavior with index  $\alpha$ . In such procedure we create the empirical cdf  $F$  and apply the regression method, namely to the function  $1 - F(x)$  we fit, by using least squares method, the power function of the form  $ax^{-\alpha}$ . As a result we obtain the tail index  $\alpha$ .

#### 4. Applications

In order to demonstrate the presented estimation methods of  $\alpha$  parameter let us first consider the simulated subdiffusion process with drift equal to zero and  $\alpha = 0.5$ . The sample trajectory is plotted in Fig 5. Because the  $\alpha$  parameter is responsible for traps observed in subordinated process before the further analysis we calculate sizes of the observed traps. As we mentioned before, the estimation procedures are based only on trap-data. In Table I we demonstrate the obtained values of the estimates of the  $\alpha$  parameter. As we observe the received values indicate the value of  $\alpha$  close to 0.5.

In the next step of our study we consider real data describing stock prices of two companies: Polish Sanwil and Estonian Kalev. Analyzed data are available on the web sites of the Polish Stock Exchange and the Nordic

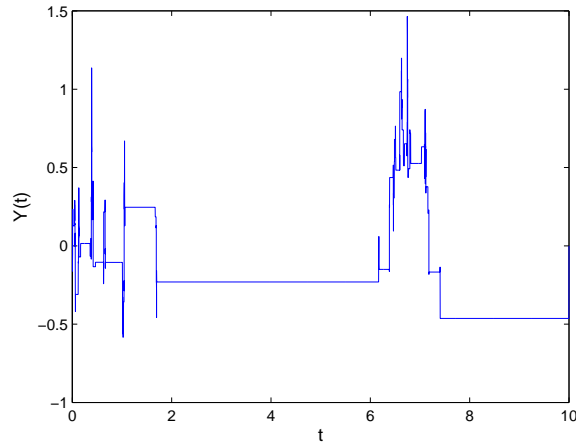


Fig. 5. Simulated trajectory of the subdiffusion process (1) with  $F = 0$  and  $\alpha = 0.5$ .

TABLE I

The values of  $\alpha$  estimates for simulated data.

$\alpha$	Hill	Pickands	EVI	POT	M-S	PCF
0.5	(0.4, 0.56)	(0.48 0.5)	(0.5, 0.6)	0.47	0.5941	0.48

Exchange Baltic Market, see [27, 28]. In Fig. 6 we demonstrate logarithmic prices of the considered assets. Similar as in simulated data we observe here the trap behavior, therefore we propose to use the subdiffusion process (1). In the analyzed logarithmic prices we do not observe any trend thus we suggest to consider the subdiffusion process with force equal to zero. Here, the constant periods of stock prices occur when the liquidity of the assets is low. It is common for emerging markets or for small company's stocks. Because each transaction on the market brings additional cost, investors usually do not pay much attention to small fluctuations of stock prices. Thus, if the price does not change more than 0.1, we treat it as a constant. The time periods of the considered stock prices are as follows: for Sanwil — from 3/10/2005 to 25/09/2008; for Kalev — from 12/02/1998 to 19/09/2008. According to the estimation procedure described at the beginning of this section first we calculate sizes of the traps. However, due to the lack of continuous data — we consider only daily prices — the obtained values are only that exceeding the time between quotations. Thus, the estimation procedures base only on this realizations of  $\alpha$ -stable random variables that are above threshold equal to 1 and characterize only it is tail behavior. However, each of the described estimates can be calibrated for using only tail observations. The Hill, the Pickands as well as the EVI procedure is



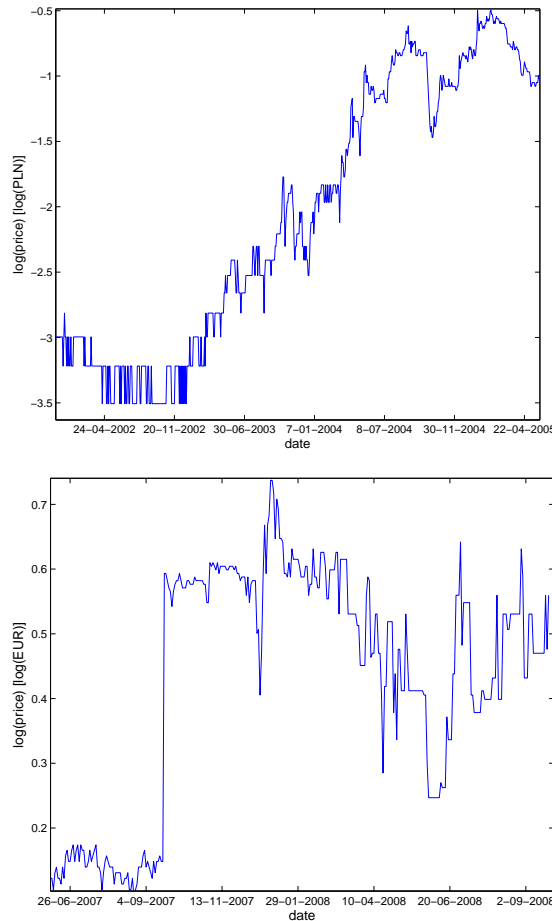


Fig. 6. Logarithmic daily prices of the Sanwil (upper panel) and the Kalev stocks (lower panel). Notice similar properties as observed in the simulated subdiffusion process, see Fig. 5. We also identify trap behavior, however constant periods are shorter than for simulation with  $\alpha = 0.5$ . It suggests higher parameter  $\alpha$ , what is consistent with the estimation results.

based only on the  $k$  last order statistics, so they are appropriate for the considered data sets. The POT estimate requires only the observations that exceed some threshold  $u$ , so for our study we choose  $u = 1$ . Also the PCF method can be calibrated to the analyzed data set by fitting the estimated non scaled part of the empirical distribution to the function  $ax^{-\alpha} + b$ . Only one from presented estimation methods, the M–S estimate, requires not only tail observations. In spite of this in order to demonstrate its behavior for such kind of data we take into consideration the values of the estimate. Notice the obtained M–S estimates are close to values of another statistics

for three considered sets of data (including simulated data). In Table II we present the obtained values of the  $\alpha$  parameter estimates. For real data we calculate the  $\alpha$  as the mean of POT, M–S and PCF statistics and then we check if the obtained values are included in the range of the Hill, Pickands and EVI estimates. Therefore, for Sanwil stock prices we obtain  $\hat{\alpha} = 0.64$  and for Kalev we get  $\hat{\alpha} = 0.86$ .

TABLE II

The values of  $\alpha$  estimates for two kinds of considered data.

Stock's name	Hill	Pickands	EVI	POT	M–S	PCF
Sanwil	(0.4, 0.9)	(0.8, 1.4)	(0.6, 0.9)	0.59	0.71	0.62
Kalev	(0.5, 0.86)	(0.6, 1)	(0.8, 0.9)	0.89	0.88	0.81

## 5. Conclusions

Many studies have shown that subdiffusion processes allow for modeling different kinds of phenomena when the diffusion systems for the observations seem not to be reasonable. The list of observed processes that exhibit subdiffusive dynamics is extensive and still growing [1–3]. We also observe such special behavior in financial data, especially in stock prices, when in some periods of time quotations exhibit small deviations or are simply on the same level.

In this paper we considered financial data in context of subdiffusion process. Such process with constant drift defined in (1) is proposed as a model describing logarithmic daily stock prices. We present the form of the solution of such subordinated system and also the explicit formulas for first and second moments as well as the covariance and correlation. The analyzed model exhibit certain deviations from the classical Brownian linear time-dependence of the centered second moment. This behavior is demonstrated by using the Monte Carlo method that is based on the link between subdiffusion and FFPE [10–13]. Moreover, we refer to the FFPE that describes the pdf of the considered process.

In this article we proposed a new approach concerning the application of the subdiffusion process to market data. This issue is still missing and, therefore, we overcome this gap by proposing estimation techniques what can be a starting point to prediction of such systems. The analyzed approach can be used in various fields connected with market concerns, such as pricing financial instruments and issues of financial engineering.

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