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Subordinated α-stable Ornstein–Uhlenbeck process as a tool for financial data description

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ABSTRACT

The classical financial models are based on the standard Brownian diffusion-type processes. However, in the exhibition of some real market data (like interest or exchange rates) we observe characteristic periods of constant values. Moreover, in the case of financial data, the assumption of normality is often unsatisfied. In such cases the popular Vasiček model, that is a mathematical system describing the evolution of interest rates based on the Ornstein–Uhlenbeck process, seems not to be applicable. Therefore, we propose an alternative approach based on a combination of the popular Ornstein–Uhlenbeck process with a stable distribution and subdiffusion systems that demonstrate such characteristic behavior. The probability density function of the proposed process can be described by a Fokker–Planck type equation and therefore it can be examined as an extension of the basic Ornstein–Uhlenbeck model. In this paper, we propose the parameters' estimation method and calibrate the subordinated Vasiček model to the interest rate data.

1. Introduction

During the past decades, continuous time models have become very popular in financial econometrics, dating back to the seminal work of Black and Scholes [1]. They are often used for pricing applications and allow one to handle irregularly spaced data in a simple and intuitive way. Moreover, they seem to be the most natural for modeling high frequency data, which became available recently, for instance in finance. One of the famous examples of continuous time models is the Ornstein–Uhlenbeck process that was originally introduced by Uhlenbeck and Ornstein [2] as a suitable model for the velocity process in the Brownian diffusion. In other words, this process provides a stationary solution for the classical Klein–Kramers dynamics [3,4]. The Ornstein–Uhlenbeck process has been of fundamental importance for theoretical studies in physics and mathematics, but it has also been used in many applications including financial data such as interest rates, currency exchange rates, and commodity prices. In finance, it is best known in connection with the Vasiček interest rate model, [5], which was one of the earliest stochastic models of the term structure. The model exhibits mean reversion, which means that if the interest rate is above the long run mean, then the drift becomes negative so that the rate will be pushed down to be closer to the mean level. Likewise, if the rate is below the long run mean, then the drift remains positive so that the rate will be pushed up to the mean level. Such mean reversion feature complies with the economic phenomenon that in the long time period interest rates appear to be pulled back to some average value.

Analysis of various real-life data shows that some processes observed in economics demonstrate characteristic behavior related to two properties. On the one hand, we observe periods in which they stay constant [6]. This feature is most common for emerging markets in which the number of participants, and thus the number of transactions, is rather low. The constant periods of financial processes correspond to the trapping events in which the subdiffusive test particle gets immobilized [7,8]. Apart from the financial applications, in the recent years systems exhibiting anomalous subdiffusive

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behavior attracted growing attention also in the various fields of physics and related sciences. The list of systems displaying subdiffusive dynamics is diverse and very extensive. It encompasses, among others, charge carrier transport in amorphous semiconductors, nuclear magnetic resonance, diffusion in percolative and porous systems, transport on fractal geometries and dynamics of a bead in a polymeric network, as well as protein conformational dynamics; see Refs. [9,10] and the references therein. On the other hand, empirical observations, especially in finance, exhibit fat tails. This heavy tailed or leptokurtic character of the distribution of price changes has been repeatedly observed in various markets; see e.g. [11] and the references therein. In response to the empirical evidence of fat tails, Mandelbrot [12] and Fama [13] proposed the α-stable distribution as an alternative to the Gaussian law. The stable distribution has found many practical applications, for instance in finance [14], physics [15–17] and electrical engineering [18]. The Ornstein–Uhlenbeck process with the α-stable distribution was analyzed in Refs. [19,20] as a suitable model for a financial data description.

In this paper, as an alternative for the classical approach, we propose a combination of the α-stable Ornstein–Uhlenbeck process and subdiffusion systems with characteristic trapping-behavior. We present the main features of the examined process and show the connections with the classical Vasiček model. Moreover, we describe the parameters' estimation procedure for the considered system and apply it to the interest rates data. Our analysis is motivated by the empirical study of data corresponding to the WIBOR (Warsaw Interbank Offered Rate), BUBOR (Budapest Interbank Offered Rate) and PRIBOR (Prague Interbank Offered Rate) rates, that exhibit two mentioned properties; see Fig. 5.

The paper is structured as follows. In Section 2, we give a definition and main properties of the classical (Gaussian), as well as, α-stable Ornstein–Uhlenbeck process and introduce the subordinated α-stable Ornstein–Uhlenbeck process. Next, in Section 3 we describe the parameters' estimation scheme for the considered model. Finally, in order to present the motivation of the paper, in Section 4 we calibrate the subordinated α-stable Ornstein–Uhlenbeck process to the interest rates data.

2. The Ornstein–Uhlenbeck processes

2.1. Ornstein–Uhlenbeck process driven by Brownian motion

The classical stationary Ornstein–Uhlenbeck (O–U) process can be obtained in two different ways. On the one hand, it is a stationary solution of a Langevin equation with a Brownian motion noise. On the other hand, it can be obtained from a Brownian motion by the so called Lamperti transformation [21]. The definition of the classical O–U process $\{X(t)\}$ with drift μ is the solution of a stochastic differential equation of the following form [2]

$$dX(t) = \theta(\mu - X(t))dt + \sigma dB(t), \tag{1}$$

where $\mu \in \mathbb{R}, \theta > 0$ and $\sigma > 0$ are the model parameters and $\{B(t)\}$ denotes the Brownian motion.

In economics, this process is called the Vasiček model [5] and is the most classical approach to modeling interest rates, currency exchange rates, and commodity prices. The popularity of that model comes from its mean-reverting feature, meaning that in a long time period the process will go back to its equilibrium level. If the process is below the (long-term) mean, the drift will be positive, pulling the process up to the equilibrium level. Analogously, if the process is above the mean, the drift will be negative, pulling the process down to the equilibrium level. The parameter μ represents the long-term mean, θ—the speed of mean-reversion, and σ—the volatility. In physical sciences, the O–U process is a prototype of a noisy relaxation process, whose probability density function (pdf) $f(x, t)$ can be described by the Fokker–Planck equation, [22–26]:

$$\frac{\partial f(x,t)}{\partial t} = \frac{\partial}{\partial x}[\theta(x-\mu)f(x,t)] + \frac{\sigma^2}{2}\frac{\partial^2 f(x,t)}{\partial x^2}. \tag{2}$$

The unique solution of Eq. (1) is the process $\{X(t)\}$ defined as follows:

$$X(t) = X(0)e^{-\theta t} + \mu(1 - e^{-\theta t}) + \sigma \int_0^t e^{-\theta(t-s)} dB(s). \tag{3}$$

The process $\{X(t)\}$ can be also represented (conditionally with given $X(0)$) as a scaled time-transformed Brownian motion:

$$X(t) = X(0)e^{-\theta t} + \mu(1 - e^{-\theta t}) + \frac{\sigma}{\sqrt{2\theta}} B\left(e^{2\theta t} - 1\right) e^{-\theta t}, \tag{4}$$

implying that $X(t)$ has a conditional (with a given $X(0)$) Gaussian distribution with mean $\mu + e^{-\theta t}(X(0) - \mu)$ and variance equal to $\frac{\sigma^2}{2\theta}(1 - e^{-2\theta t})$. For a constant $X(0)$ this distribution tends, as $t \to \infty$, to the stationary distribution $N\left(\mu, \frac{\sigma^2}{2\theta}\right)$ [27,28], so differently from the Brownian motion, the O–U process admits a bounded variance.

One can find the main properties of the O–U process in Ref. [29]. Note that the simulation and estimation procedures for the process $\{X(t)\}$ given in (3) can be based on the Euler scheme discretization:

$$X(t + \Delta t) - X(t) = \theta(\mu - X(t))\Delta t + \sigma\sqrt{\Delta t} B_t, \tag{5}$$

where $B_t = B(t + \Delta t) - B(t),\ t = 0, \Delta t, 2\Delta t, \ldots$ constitutes a sequence of independent and identically distributed (i.i.d.) random variables following the standard normal distribution and Δt is the discretization step. Observe that for $\Delta t = 1$ Eq. (5) defines a discrete-time autoregressive model of order 1 (AR(1)) [30]. A sample trajectory of the process defined by (1) with parameters $\theta = 0.5$, $\mu = 0$, $\sigma = 4$ and $X(0) = 5$ is presented in Fig. 1.

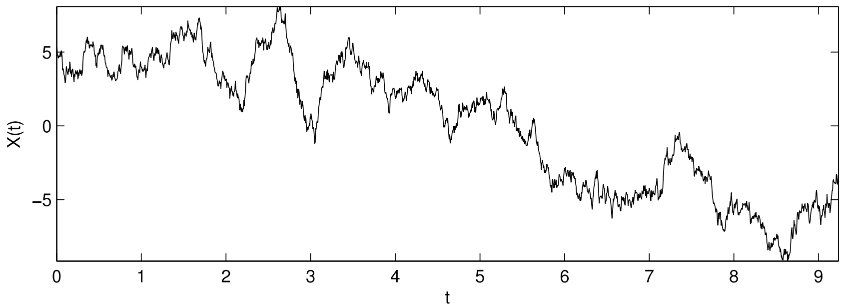

Fig. 1. A sample trajectory of the classical O–U process (see Eq. (1)) with parameters $\theta = 0.5$, $\mu = 0$, $\sigma = 4$ and $X(0) = 5$.

2.2. Ornstein–Uhlenbeck process driven by α-stable Lévy motion

The standard approach in modeling asset prices is to assume that the analyzed data follows a Gaussian distribution. This is also the case for the classical Vasiček model, in which the described asset dynamics is given by the O–U process defined in Eq. (1). Unfortunately, the assumption of normality is often unsatisfied. Many studies have shown that heavy-tailed distributions yield more reliable than the Gaussian results when modeling different kinds of phenomena [31–35]. Especially the α-stable distributions [36] have found many practical applications, for instance in finance [14], physics [15] and electrical engineering [18]. It is interesting to note that, besides the lack of normality, financial data often exhibit power-law autocorrelations in volatility [37,38] as well as power-law cross-correlations in volatility [39]. For the interest rates datasets analyzed in this paper (see Fig. 5), the sizes of price changes indicate that a distribution with heavier tails than the Gaussian dynamics should be used. A possible approach to modeling such processes is to replace the Brownian motion in representation (1) with the α-stable Lévy one. In this case, the tails of the analyzed process are asymptotically equivalent to the Pareto law, i.e. they exhibit the power-law behavior, [11]. The O–U process with an α-stable distribution was analyzed in Ref. [19] as a suitable model for a financial data description.

The O–U process driven by the α-stable Lévy motion is a unique solution of the following stochastic differential equation

$$dX_\alpha(t) = \theta(\mu - X_\alpha(t))dt + \sigma dZ(t), \tag{6}$$

where $\{Z(t)\}$ is a Lévy process with symmetric α-stable increments. For clarity, we assume that the scale parameter of the process $\{Z(t)\}$ is equal to 1. This process was analyzed in detail in Refs. [40–42] as a special case of the considered continuous-time ARMA models. The pdf $f(x,t)$ of the process $\{X_\alpha(t)\}$ for a given $X_\alpha(0)$ can be described, similarly as in the Gaussian case, by the Fokker–Planck-type equation of the following form [43]:

$$\frac{\partial f(x,t)}{\partial t} = \frac{\partial}{\partial x}(\theta(x-\mu)f(x,t)) + \sigma^\alpha \nabla^\alpha f(x,t), \tag{7}$$

where ∇^α, $0 < \alpha \leq 2$, is the Riesz fractional derivative [44].

Similarly as in the Gaussian case, the simulation procedure for the process $\{X_\alpha(t)\}$ is based on the Euler scheme discretization of Eq. (6):

$$X(t+\Delta t) - X(t) = \theta(\mu - X(t))\Delta t + \sigma(\Delta t)^{\frac{1}{\alpha}} Z_t, \tag{8}$$

where $Z_t = Z(t+\Delta t) - Z(t)$, $t = 0, \Delta t, 2\Delta t, \ldots$ is a sequence of i.i.d. random variables with symmetric α-stable distribution with the scale parameter equal to 1 and Δt is the discretization step. A sample trajectory of the stable O–U process with parameters $\theta = 0.5$, $\mu = 0$, $\sigma = 4$, $\alpha = 1.5$ and $X_\alpha(0) = 5$ is given in Fig. 2. Observe, contrarily to the trajectory of the Gaussian O–U process (see Fig. 1), large jumps of the process values, indicating non-Gaussian dynamics.

2.3. The subordinated stable Ornstein–Uhlenbeck process

The stable Lévy motion in representation (6) allows for modeling processes with heavy-tailed dynamics. However, as we observe for instance in Fig. 5, the analyzed data exhibit not only heavy-tailed changes but also some periods of constant values. Obviously, such behavior cannot be appropriately described by the standard Langevin equation (6). Hence, we propose to use a stable Ornstein–Uhlenbeck process (6) combined with the subordination scenario. Precisely, the subordinated stable O–U process is defined as [45]:

$$Y(t) = X_\alpha(S_\gamma(t)), \tag{9}$$

where $X_\alpha(\tau)$ is given by (6). $\{S_\gamma(t)\}_{t\geq 0}$ is the inverse γ-stable subordinator, defined as

$$S_\gamma(t) = \inf\{\tau > 0 : U_\gamma(\tau) > t\}, \tag{10}$$

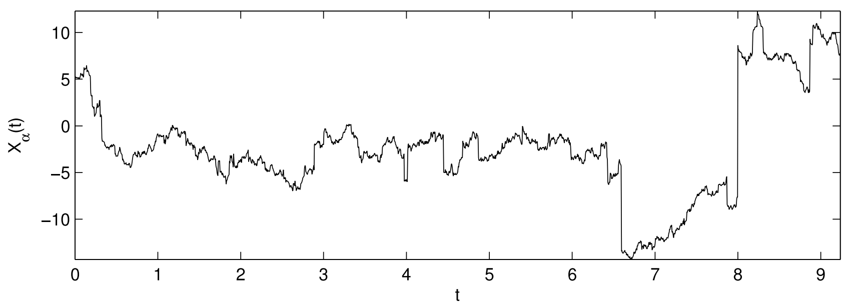

Fig. 2. A sample trajectory of the α-stable O–U process (see Eq. (6)) with parameters $\theta = 0.5, \mu = 0, \sigma = 4, \alpha = 1.5$ and $X_\alpha(0) = 5$. Observe large jumps of the process values, indicating non-Gaussian dynamics.

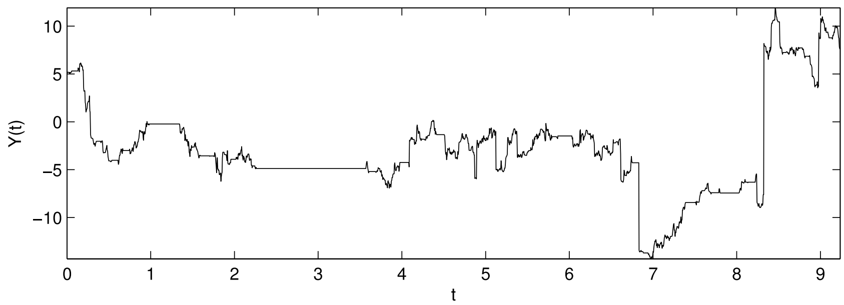

Fig. 3. A sample trajectory of the subordinated α-stable O–U process (see Eq. (9)) with parameters $\theta = 0.5, \mu = 0, \sigma = 4, \alpha = 1.5, \gamma = 0.9$ and $X_\alpha(0) = 5$. Observe periods of constant values, indicating a subordination scenario.

where $\{U_\gamma(\tau)\}_{\tau\geq 0}$ is a γ-stable nondecreasing Lévy process [46] with the Laplace transform $E(e^{-uU_\gamma(\tau)}) = e^{-\tau u^\gamma}, 0 < \gamma < 1$. The probability distribution function of the process (9) is given by the fractional Fokker–Planck equation of the form [45]:

$$\frac{\partial f(x,t)}{\partial t} = {}_0D_t^{1-\gamma}\left[\frac{\partial}{\partial x}(\theta(x-\mu)f(x,t)) + \sigma^\alpha \nabla^\alpha f(x,t)\right], \tag{11}$$

where the fractional derivative of the Riemann–Liouville type is defined as [44]:

$${}_0D_t^{1-\gamma}f(t) = \frac{1}{\Gamma(\gamma)}\frac{d}{dt}\int_0^t (t-s)^{\gamma-1}f(s)ds \tag{12}$$

for $f \in C^1([0,\infty))$ and $\gamma \in (0,1)$.

A detailed description of the simulation procedure for the subordinated Langevin equations with an α-stable noise one can find in Ref. [45]. A sample trajectory of the subordinated stable O–U process defined by (9) with parameters $\theta = 0.5$, $\mu = 0, \sigma = 4, \alpha = 1.5, \gamma = 0.9$ and $X_\alpha(0) = 5$ is plotted in Fig. 3. In order to illustrate the effect of subordination, the trajectory is generated for the same realization of the process $X_\alpha(\tau)$ as in Fig. 2.

3. The estimation procedure

Assume that the observed time series $\mathbb{Y} = (Y(t_1), Y(t_2), \ldots, Y(t_n))$ follows the subordinated α-stable O–U process given by Eq. (9). The estimation procedure for the parameters $\gamma, \mu, \theta, \sigma$ and α can be divided into three steps. First, the subordinator $S_\gamma(t)$ and the stable O–U process $X_\alpha(\tau)$ are separated. Next, the parameter γ is computed from the data corresponding to the process $S_\gamma(t)$. Finally, the parameters of the process $X_\alpha(\tau)$ are estimated based on a discretization of the Langevin equation (6). The detailed estimation algorithm is illustrated in Fig. 4 and described in what follows.

Decomposition. Following Ref. [47], $\mathbb{Y}$ is decomposed into two separate vectors—$\mathbb{E}$ corresponding to the subordinator $S_\gamma(t)$ and $\mathbb{X}$ corresponding to the Langevin equation $X_\alpha(t)$. The elements of the vector $\mathbb{E} = (E_1, E_2, \ldots, E_{n_1})$ are equal to the lengths of constant periods of $\mathbb{Y}$. Precisely, $E_j = t_i - t_{i-k}$ if for all $m \in \{0, 1, 2, \ldots, k\} Y(t_i) = Y(t_{i-m}), Y(t_i) \neq Y(t_{i+1})$ and $Y(t_i) \neq Y(t_{i-k-1})$. The vector $\mathbb{X} = (X_1, X_2, \ldots, X_{n_2})$ is build from the values of $\mathbb{Y}$ corresponding to the non-constant periods, namely $X_j = Y(t_i)$ if $Y(t_i) \neq Y(t_{i-1})$.

Estimation of the parameter γ. Observe that the lengths of the constant periods of $Y(t)$ are equal to the jumps of the process $U_\gamma(\tau)$, (10). Hence, if the process $Y(t)$ was observed in all time points, the vector $\mathbb{E}$ would consist of an i.i.d. sample following totally skewed γ-stable distribution. The parameter γ could than be obtained with one of the standard estimation

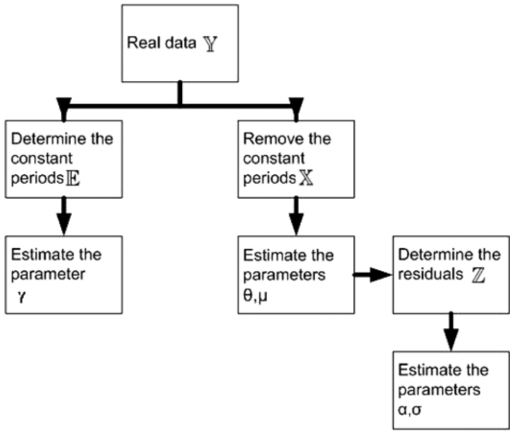

Fig. 4. The estimation scheme for the derivation of the parameters $\gamma, \mu, \theta, \sigma$ and α for the subordinated α-stable O–U process.

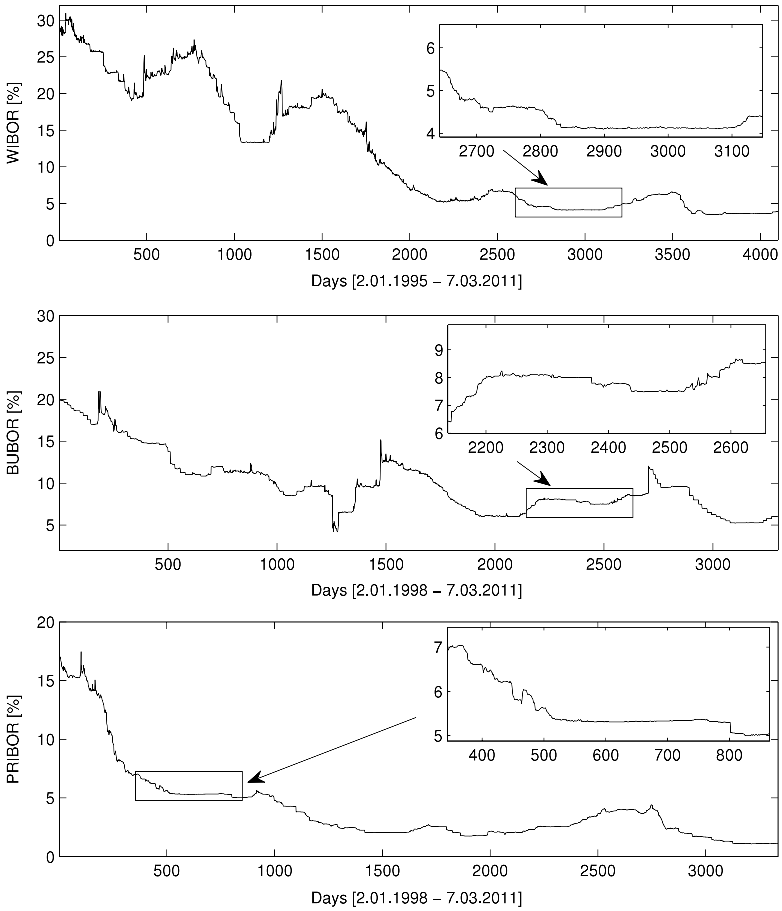

Fig. 5. The examined datasets (vector $\mathbb{Y}$) of the 1-monthly WIBOR (top panel), BUBOR (middle panel) and PRIBOR (bottom panel) rates. Observe the trapping-behavior, as well as, the heavy-tail changes, typical for the subordinated processes with an α-stable structure.

procedures for the stable distribution. However, the analyzed data is usually reported in discrete equally spaced time steps (e.g. for daily data we have $t_i = i$) and, hence, the jumps shorter than the discretization step are not observed. Therefore, we propose to use a method based on the behavior of the upper tail of the stable distribution that in this case is of the form $1-F(x) \sim x^{-\gamma}$. In the considered estimation procedure, we construct the empirical cumulative distribution function F_n and apply the regression method. Namely, by using the least squares method, we fit a power function to the empirical upper tail $1 - F_n(x)$. As a result we obtain the stability index γ; see [6].

Table 1
Estimates of the subordinated α-stable O–U process parameters calculated from 100 simulated trajectories. The values in brackets represent 95% confidence intervals for the received values.

$\hat{\gamma}$	$\hat{\mu}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\sigma}$
$0.7798^{(0.6330,0.8557)}$	$13.9858^{(13.8299,14.1509)}$	$0.1172^{(0.0748,0.2854)}$	$1.5006^{(1.3720,1.7009)}$	$0.04^{(0.0329,0.0478)}$

Table 2
Estimates of the γ parameter obtained for the three analyzed datasets. The values in brackets represent 95% confidence intervals.

	WIBOR	BUBOR	PRIBOR
$\hat{\gamma}$	$0.9728^{(0.9232,1.022)}$	$0.6274^{(0.5974,0.6574)}$	$0.6869^{(0.5876,0.7862)}$

Ornstein–Uhlenbeck process parameters' estimation. In order to estimate the parameters of the process $X_\alpha(\tau)$ we use the concept of discretization (with the discretization step $\Delta t = 1$) of the α-stable O–U process represented by Eq. (6). Observe that in this case the time series $\mathbb{X} = (X_1, X_2, \ldots, X_{n_2})$ obtained from $\mathbb{Y}$'s decomposition is given by the process defined in (8), which for $\Delta t = 1$ is an α-stable AR(1) model. The estimation procedure starts with μ and θ derivation. Observe, that AR(1) model is a special case of ARMA(p, q) system with orders $p = 1$ and $q = 0$. Therefore, the parameters μ and θ can be estimated by using the method described in Ref. [48]. This procedure, the so-called Whittle estimator, is based on the sample periodogram of the analyzed time series. Once the μ and θ estimates are found, the residuals of the process can be derived. From (8) the residuals series $\mathbb{Z} = (Z_1, Z_2, \ldots, Z_{n_2-1})$, where $Z_t = X_t - (1-\theta)X_{t-1} - \theta\mu$ yields an i.i.d. symmetric α-stable sample. Now, the parameters α and σ can be estimated from the vector $\mathbb{Z}$ using e.g. the regression or the maximum likelihood method; see Ref. [11] for details. The α parameter, representing the stability index of the Lévy distribution, can be also estimated with one of the quantile methods, like Hill's estimator, or a method based on the detrended fluctuation analysis (DFA); see Ref. [49] and the references therein. Note that, if $\Delta t \neq 1$, the estimates of θ and σ parameters should be calculated as $\hat{\theta} = \hat{\theta_1}/\Delta t$ and $\hat{\sigma} = \hat{\sigma_1}/\Delta t^{1/\hat{\alpha}}$, where $\hat{\theta_1}$ and $\hat{\sigma_1}$ are the estimates for suitable parameters obtained by the assumption $\Delta t = 1$. The other parameters are not sensitive to the discretization step.

In order to verify the estimation procedure we simulate 100 trajectories of the subordinated α-stable O–U process with the following parameters:

$$\gamma = 0.8, \qquad \mu = 14, \qquad \theta = 0.1, \qquad \alpha = 1.5, \qquad \sigma = 0.04.$$

Each datasets is represented by 5000 observations. On the basis of the simulated data, we estimate the γ, μ, α and σ parameters by using the presented procedure. In Table 1, we present the medians of the obtained estimates and the 95% confidence intervals for the received values.

4. Data analysis

In order to present the motivation for introducing the subordinated α-stable O–U process with stable waiting-times, we examine the real financial data of the monthly WIBOR (Warsaw Interbank Offered Rate), BUBOR (Budapest Interbank Offered Rate) and PRIBOR (Prague Interbank Offered Rate). The data is quoted daily and comes from the period 02.01.1995–07.03.2011 for WIBOR (4101 observations) or 02.01.1998–07.03.2011 for BUBOR and PRIBOR (3299 observations). The interbank rates are calculated from the rates offered by the major banks in the corresponding country. A monthly rates refer to the interest rate on interbank loans of one month. The examined datasets are presented in Fig. 5. Observe the heavy-tailed changes of the interest rates, as well as, the periods of constant values. Motivated by both of these features, we propose to use the subordinated stable O–U process with the stable subordinator as an appropriate model describing these financial time series.

According to the estimation procedure presented in Section 3, we divide the examined datasets into two vectors. The first one, vector $\mathbb{E}$, represents the lengths of constant periods of the examined time series $\mathbb{Y}$ (see Fig. 6), while the second, vector $\mathbb{X}$, is built from the values of $\mathbb{Y}$ corresponding to the non-constant periods. After the decomposition, we estimate the parameter γ, i.e. the index of stability of the subordinator's distribution, using the method described in the previous section. The obtained values of the estimates are given in Table 2. Recall that the value of γ in the γ-stable subordinator is assumed to be lower than 1, $\gamma < 1$. However, the value of the γ parameter obtained for WIBOR dataset is close to 1 and the 95% confidence interval contains 1. This may indicate that some other subordinator, e.g. tempered stable (see Ref. [50]), would be more appropriate for this dataset. This is not the case for the BUBOR and PRIBOR datasets, while the values of γ are well below 1.

In the next step we estimate the parameters of the α-stable O–U process, that is represented by the vector $\mathbb{X}$. First, we calculate the estimates of the parameters μ and θ. The estimation is based on the discrete AR(1) version of the stable O–U process, given in (8) with $\Delta t = 1$. Second, using the maximum likelihood method, we derive the values of the parameters α and σ (i.e. the indexes of stability and scale, respectively) of the residuals' distribution (see Fig. 7 for residual series plot). The

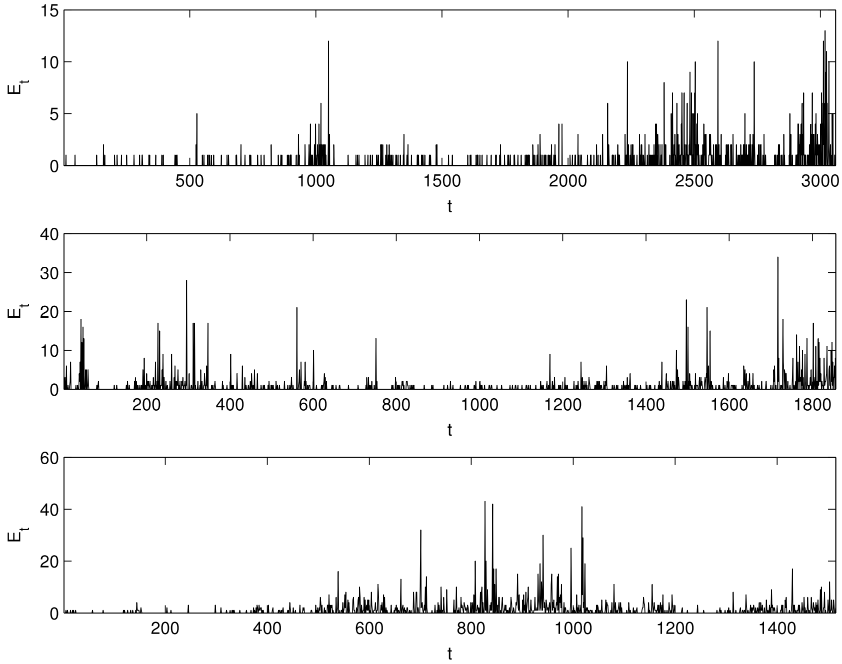

Fig. 6. The lengths of constant periods (vector $\mathbb{E}$) for the WIBOR (top panel), BUBOR (middle panel) and PRIBOR (bottom panel) data.

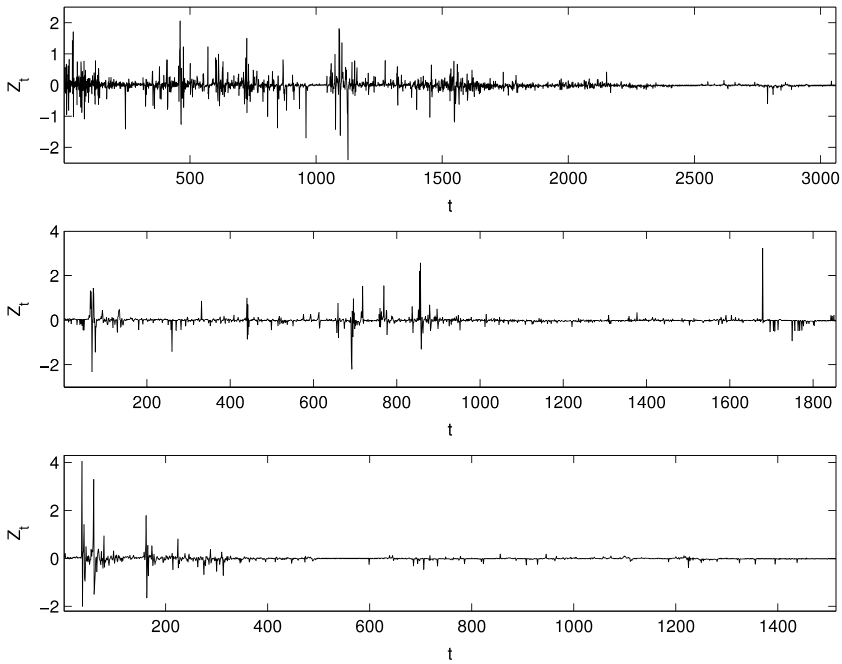

Fig. 7. The residual series (vector $\mathbb{Z}$) obtained from the α-stable O–U process fitted to the WIBOR (top panel), BUBOR (middle panel) and PRIBOR (bottom panel) data.

obtained values are given in Table 3. Recall, that the parameter θ is responsible for the speed of mean reversion. Here, for all datasets we obtain similar, rather slow, rate of order 10^{-3}. The parameter μ represents the long-term mean of the process, while σ is the scale parameter of the residual distribution. The values of obtained α estimate, being the tail index of the stable distribution, indicate heavy tailed distribution for all datasets. We have also estimated the values of the α parameter with Hill's method and the DFA method. The following values were obtained: $\alpha_{DFA} = 1.12, \alpha_{Hill} \in (1.1, 1.3)$ (WIBOR); $\alpha_{DFA} = 1.29, \alpha_{Hill} \in (1, 1.15)$ (BUBOR); $\alpha_{DFA} = 1.10, \alpha_{Hill} \in (1.08, 1.18)$ (PRIBOR).

In order to verify the assumption on symmetry of residuals, we have also estimated the skewness and mean parameters, but the obtained values in all cases were very close to 0.

Table 3
Estimates of the α-stable O–U process parameters obtained for the three analyzed datasets.

	WIBOR	BUBOR	PRIBOR
$\hat{\mu}$	13.5404	10.1117	5.8871
$\hat{\theta}$	0.0017	0.0060	0.0034
$\hat{\alpha}$	1.1261	1.2453	1.1677
$\hat{\sigma}$	0.0352	0.0277	0.0180

Table 4
Values of the goodness-of-fit statistics calculated for the vector of residuals ($\mathbb{Z}$) obtained from four versions of the Ornstein–Uhlenbeck process.

	Anderson–Darling	Kolmogorov–Smirnov	Kuiper	Cramér–von-Mises
	WIBOR			
Gaussian O–U	Inf	17.29	32.36	120.68
Subordinated Gaussian O–U	Inf	13.49	24.88	70.57
Stable O–U	103.716	10.76	12.08	23.11
Subordinated stable O–U	4.72	2.43	4.72	0.84
	BUBOR			
Gaussian O–U	Inf	19.21	36.70	140.64
Subordinated Gaussian O–U	Inf	12.80	24.43	62.28
Stable O–U	29.96	5.55	6.66	6.18
Subordinated stable O–U	2.01	1.36	2.69	0.28
	PRIBOR			
Gaussian O–U	Inf	21.48	42.10	184.41
Subordinated Gaussian O–U	Inf	12.78	24.31	65.36
Stable O–U	243.02	13.96	16.13	54.58
Subordinated stable O–U	7.16	2.42	4.67	1.15

Finally, in order to show the benefits from introducing the subordinated α-stable Ornstein–Uhlenbeck process we calculate the goodness-of-fit statistics for the residual distribution in four types of the Ornstein–Uhlenbeck process, i.e. the classical Gaussian (1), the subordinated Gaussian, the α-stable (6) and the subordinated α-stable one (9). We use the Anderson–Darling, Kolmogorov–Smirnov, Kuiper and Cramér–von-Mises goodness-of-fit tests [51], that are based on the supremum distance statistic between the empirical and theoretical distribution function. All obtained values (see Table 4) clearly show that the introduction of the subordination scenario, as well as, the α-stable distribution leads to the significant improvement in the fit of the residuals (recall that the lower is the test statistic the better is the fit).

5. Conclusions

In this paper we have analyzed an α-stable Ornstein–Uhlenbeck process with stable waiting-times. We have reviewed the main features of the examined system and presented the parameters' estimation procedure. Moreover, we have calibrated the subordinated α-stable Ornstein–Uhlenbeck process to three datasets of the interbank rates.

The proposed approach is an alternative for the classical Vasiček interest rates model. It reflects not only the heavy-tailed behavior of the analyzed time series, but also allows for modeling constant periods reported in the data. The latter is observed in the interest rates data but is also important when modeling prices in emerging markets. The low liquidity of such markets results in apparent constant periods [6]. Moreover, the subordinated process might be useful in modeling high-frequency (especially tick-by-tick) data, when the waiting times between the individual transactions are observed.

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