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Scale-invariance of random populations: From Paretian to Poissonian fractality

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Abstract

Random populations represented by stochastically scattered collections of real-valued points are abundant across many fields of science. Fractality, in the context of random populations, is conventionally associated with a Paretian distribution of the population's values.

Using a Poissonian approach to the modeling of random populations, we introduce a definition of "Poissonian fractality" based on the notion of scale-invariance. This definition leads to the characterization of four different classes of *Fractal Poissonian Populations*—three of which being non-Paretian objects. The Fractal Poissonian Populations characterized turn out to be the unique fixed points of natural renormalizations, and turn out to be intimately related to Extreme Value distributions and to Lévy Stable distributions.

Keywords: Fractals; Scale-invariance; Renormalization; Paretian fractality; Poisson processes; Poissonian fractality; Extreme value distributions; Lévy Stable distributions

1. Introduction

In the past few decades the exploration and application of fractal geometry have drawn tremendous interest across a broad array of scientific fields [1–5] including, in particular, the physical sciences [6–9]. The characteristic geometric feature of fractal objects is *invariance under changes of scale*.

Deterministic scale-invariance is well exemplified by "structural fractals" such as Cantor's set [10], Koch's curve [11], and Sierpinski's gasket [12]. Structural fractals are generically composed of pieces—each of which "looks" exactly like the whole object.

Statistical scale-invariance is well exemplified by "self-similar processes" such as Brownian motion [13], Lévy motions [14,15], and fractional stable motions [16–18]. Self-similar processes are random motions which, statistically, "look the same" when zooming-in and zooming-out on their sample-path trajectories [19].

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1.1. Random populations and Paretian fractality

This paper is devoted to the study of fractality in the context of *random populations*: populations represented by collections of points scattered randomly on the real line (or on a part of it).

Examples of random populations include: earthquakes taking place in a given geological region, during a given period of time, measured by their magnitudes—each point representing the magnitude of an earthquake; stars in a given sector of space measured by their masses—each point representing the mass of a star; citizens of a given state measured by their wealth—each point representing the wealth of a citizen; insurance claims in a given insurance-portfolio measured by their costs—each point representing the cost of a claim, etc.

Random populations are discrete objects with no natural geometry. The lack of geometry renders unclear the very definition of scale-invariance. Fractality, in this case, is defined *algebraically* rather than geometrically: a random population is said to be *fractal* if its population-values and their occurrence-frequencies are connected via a *power-law* (see, for example, [1, Chapter 38]). Such a power-law statistical connection is referred to as *Paretian*—named after the Italian economist Vilfredo Pareto who discovered, in 1909, a power-law distribution of wealth in human societies [20].

The Paretian definition of fractality is thus based on two foundations: the transformation of a given random population into the empirical distribution of its population-values and the association of fractality with power-laws. Underlying these foundations, respectively, are the following implicit assumptions: (i) random populations are characterized by probability distributions; (ii) scale-invariance is characterized by power-laws. In what follows we challenge and question the validity of both these assumptions.

1.2. Poissonian modeling and Poissonian fractality

Rather than transforming random populations into their corresponding empirical distributions we treat them as is—collections of stochastically scattered real-valued points. The mathematical "toolbox" for the modeling and analysis of random scattering of points on general domains is the theory of *Poisson point processes* [21].

In a recently published book, Lowen and Teich comprehensively explore a deep interplay between fractals and Poisson point processes [22]. In this paper we use Poisson point processes in order to model random populations, and in order to define fractality in the context of infinite such populations.

The Poissonian modeling approach splits random populations into two exclusive categories: finite and infinite. Finite Poissonian populations are single-scale objects representable by probability distributions. Infinite Poissonian populations, on the other, are multi-scale objects which are *not* representable by probability distributions. The latter category of random populations defies the first implicit assumption underlying the Paretian definition of fractality.

The Poissonian modeling approach enables us to "go back to first principles" and define fractality via the elemental notion of scale-invariance—rather than via the implied notion of power-laws. This Poissonian definition of fractality, in turn, leads to the characterization of four different classes of Fractal Poissonian Populations:

(I) *Real Poissonian Fractals*—corresponding to populations with unbounded values, and governed by an *exponential structure*; (II) *Positive Poissonian Fractals*—corresponding to populations with values bounded from below, and governed by a *power-law structure* (with negative exponents); (III) *Negative Poissonian Fractals*—corresponding to populations with values bounded from above, and governed by a *power-law structure* (with positive exponents); (IV) *Unitary Poissonian Fractals*— corresponding to populations with bounded values, and governed by a *logarithmic structure*.

The Real and Unitary fractal classes defy the second implicit assumption underlying the Paretian definition of fractality.

1.3. Renormalization, Extreme Value distributions, and Lévy Stable distributions

Fractality is often intertwined with *renormalization*. Investigating the renormalization of Poissonian populations, we further present a "renormalization characterization" of the four aforementioned fractal classes:

(I) *Real Poissonian Fractals* are the unique fixed points of *translational renormalizations* on the real line; (II) *Positive Poissonian Fractals* are the unique fixed points of *multiplicative renormalizations* on the positive half-line; (III) *negative Poissonian Fractals* are the unique fixed points of *multiplicative renormalizations* on the negative half-line; (IV) *Unitary Poissonian Fractals* are the unique fixed points of *power-law renormalizations* on the unit interval.

The first three fractal classes are "linear" in the sense that their characterizing renormalizations—translational and multiplicative—are linear. This intrinsic linearity is the source of an intimate connection between Real, Positive, and Negative Poissonian Fractals and *Extreme Value distributions*, and between Positive Poissonian Fractals and *Lévy Stable distributions*.

Extreme Value Theory and Fluctuation Theory study the asymptotic statistical behavior of, respectively, the linearly scaled maxima and the linearly scaled sums of sequences of independent and identically distributed random variables. The resulting probabilistic limit laws are Extreme Value distributions—in the case of maxima [23,24]—and Lévy Stable distributions—in the case of sums [25,26]. These probabilistic limit laws are, in fact, sum and maximum "projections" of the three "linear" classes of Fractal Poissonian Populations described above.

The Unitary fractal class, on the other hand, is nonlinear—due to its nonlinear power-law characterizing renormalization. This intrinsic nonlinearity renders the sum and maximum "projections" of Unitary Poissonian Fractals beyond the realm of Extreme Value Theory and Fluctuation Theory.

1.4. Organization

The manuscript is organized as follows: We begin, in Section 2, with a mathematical description of Poisson-modeled random populations. In Section 3, a *definition* of Fractal Poissonian Populations—based on the notion of scale-invariance—is introduced, followed by an explicit *characterization* of these populations. Section 4 considers the *renormalization* of Poissonian populations, and studies the intimate relationship between the operation of renormalization and the Poissonian definition of fractality. Section 5 explores the *maxima* of Poissonian populations, and the connection between Fractal Poissonian Populations and *Extreme Value distributions*. Section 6 explores the *sums* of positive-valued Poissonian populations, and the connection between fractal such populations and *Lévy Stable distributions*.

1.4.1. A note about nomenclature and notation

IID = independent and identically distributed; PDF = probability density function; and $F^{-1}(\cdot)$ denotes the inverse of a real invertible function $F(\cdot)$.

2. Random populations

Consider a random population $\mathscr{P}$ represented by a collection of real-valued points scattered stochastically across the range $\mathscr{R} = (L, U)$, where L ($L \geqslant -\infty$) is the range's lower bound and U ($U \leqslant \infty$) is the range's upper bound. The points represent some real-valued physical measure of the population's members, e.g., wealth, height, mass, energy, etc.

2.1. The IID and the Poissonian algorithms

The most common way of constructing a random population $\mathscr{P}$ is the "IID algorithm": (i) determine the size N of the population; and, (ii) generate N IID random variables $\{X_1, \dots, X_N\}$, drawn from a probability distribution on the range $\mathscr{R}$ with PDF $f(\cdot)$. The random variable X_n represents the value of the nth member of the population ($n = 1, \dots, N$).

The population size N can be either deterministic or an integer-valued random variable. The PDF $f(\cdot)$ governs the probability distribution of a *generic member* of the population.

A different and less common way of constructing a random population $\mathscr{P}$ is the "*Poissonian algorithm*": generate an inhomogeneous Poisson process on the range $\mathscr{R}$ with Poissonian rate function $r(\cdot)$ [21]. The points of the Poisson process represent the values of the population's members.

Informally, the Poissonian construction means that the infinitesimal interval $(x, x + \mathrm{d}x)$ is empty with probability $1 - r(x)\,\mathrm{d}x$, and contains a single point with probability $r(x)\,\mathrm{d}x$ (independent of all other infinitesimal intervals). Rigorously, the Poissonian construction means that: (i) the number of points residing in the interval $I \subset \mathscr{R}$ is Poisson-distributed with mean $\int_I r(x)\,dx$; and (ii) the number of points residing in disjoint intervals are independent random variables.

Random populations constructed by the Poissonian algorithm shall henceforth be referred to as "Poissonian populations".

2.2. Poissonian sub-populations

Given a Poissonian population $\mathscr{P}$, set $\mathscr{P}(l) := \mathscr{P} \cap (l, U)$ to be the Poissonian sub-population of points located above the level l ($l \in \mathscr{R}$). The size $N(l)$ of the Poissonian sub-population $\mathscr{P}(l)$ is Poisson-distributed with mean

$$R(l) := \int_l^U r(x)\,dx. \tag{1}$$

We assume that $R(l)$ is finite for all levels $l \in \mathscr{R}$, but not necessarily for the lower bound level $l = L$. That is, we assume that there are only *finitely* many members of the Poissonian population $\mathscr{P}$ above any given level $l \in \mathscr{R}$. The size of the entire Poissonian population $\mathscr{P}$, however, may be either finite (if $R(L) < \infty$) or infinite (if $R(L) = \infty$).

Note that the function $R(\cdot)$ decreases monotonically from the level $R(L) \leqslant \infty$ to the level $R(U) = 0$. Moreover, the function $R(\cdot)$ *characterizes* the Poissonian population $\mathscr{P}$. The function $R(\cdot)$ is henceforth referred to as the population's "Poissonian tail function".

An immediate consequence of the "existence theorem" of the theory of Poisson processes (see [21, Section 2.5]) is the following result:

Proposition 1. *The Poissonian sub-population $\mathscr{P}(l)$ ($l \in \mathscr{R}$) admits the representation*

$$\mathscr{P}(l) = \{X_1(l), \ldots, X_{N(l)}(l)\}, \tag{2}$$

where $\{X_1(l), X_2(l), \ldots\}$ are IID random variables defined on the sub-interval (l, U) (and independent of the random variable $N(l)$), governed by the survival probability

$$\mathbf{P}_>(x) = \frac{R(x)}{R(l)} \quad (l \leqslant x < U). \tag{3}$$

In other words, Proposition 1 implies that the Poissonian sub-population $\mathscr{P}(l)$ ($l \in \mathscr{R}$) can be generated by the IID algorithm with size $N(l)$ and generic member $X_1(l)$.

2.3. Finite and infinite random populations

In the case of the IID algorithm, a finite random population corresponds to a finite size $N < \infty$, whereas an infinite random population corresponds to an infinite size $N = \infty$. Be the random population finite or infinite—it is characterized by the *same* generic member. That is, the distribution of the population's generic member is independent of the population's size. In the case of the Poissonian algorithm, however, the situation is profoundly different:

- *Finite populations*. A finite Poissonian population corresponds to a finite overall Poissonian rate $R(L) < \infty$. In this scenario Eqs. (2)–(3) hold also for the lower bound level $l = L$, the IID algorithm and the Poissonian algorithm are equivalent, and the population is characterized by the distribution of its generic member—the random variable $X_1(L)$.
- *Infinite populations*. An infinite Poissonian population corresponds to an infinite overall Poissonian rate $R(L) = \infty$. In this scenario Eqs. (2)–(3) do *not* hold for the lower bound level $l = L$ and, consequently, there is no single generic member which characterizes the entire population. Infinite Poissonian populations are thus *multi-scale objects* which lack the notion of a representative generic member.

The Poissonian algorithm keenly allows what the IID algorithm strictly prohibits: non-integrability. The IID algorithm is based on the notion of PDFs—which, by definition, are integrable (and normalized). The Poissonian algorithm, on the other hand, is based on rate functions—which may or may not be integrable.

In the case of integrable Poissonian rate functions the two algorithms coincide: the Poissonian rate function, up to a normalizing constant, equals the PDF. The case of non-integrable Poissonian rate functions, however, has no "IID counterpart": an infinite Poissonian population *cannot* be described by a PDF of a single generic member.

Poissonian populations thus split into two exclusive categories: finite and infinite. Finite Poissonian populations are single-scale objects, representable by a generic member, which can be generated by the IID algorithm. Infinite Poissonian populations, on the other, are multi-scale objects, *not* representable by a single generic member, which lie beyond the realm of the IID algorithm.

2.4. Paretian random populations

In 1909 the Italian economist Vilfredo Pareto came up with a dramatic empirical discovery. Having analyzed reams of income-data, Pareto concluded that the distribution of wealth in human societies is governed by *power-laws* [20]. Specifically, Pareto asserted that the frequency of individuals whose wealth is greater than a given level l is well approximated by a power-law of the form $al^{-\alpha}$ (the coefficient a and the exponent α being positive parameters).

Let us examine the construction—consistent with Pareto's findings—of a random population model representing the wealth of individuals.

In the case of the IID algorithm, Pareto's discovery implies that the wealth-distribution of the population's generic member should be taken to be governed by the survival probability $\mathbf{P}_{>}(x) = ax^{-\alpha}$. However, since the power-law $ax^{-\alpha}$ diverges at the origin—a *cutoff* must be introduced. Taking the resolution level l ($l>0$) to be the cutoff yields the Paretian survival probability $\mathbf{P}_{>}(x) = (x/l)^{-\alpha}$ ($x \geqslant l$).

In the case of the Poissonian algorithm, Pareto's discovery implies that the average number of individuals whose wealth is greater than a given level l should be taken to be governed by the Poissonian tail function $R(l) = al^{-\alpha}$. In this case—contrary to the case of the IID algorithm—the divergence of the power-law $al^{-\alpha}$ at the origin causes no problem, and no cutoff need be introduced.

The IID and the Poissonian algorithms follow different interpretive approaches: the former considers the *probability* of a *generic individual* to be wealthier than a given level, whereas the latter considers the *average number* of individuals wealthier than a given level. The IID algorithm implicitly assumes the existence of a generic population member, and is based on PDFs—which must be normalized. The Poissonian algorithm, on the other hand, does not require the existence of a generic population member, and is based on rate functions—which are not constrained by integrability and normalization.

Moreover, substituting the power-law Poissonian tail function $R(l) = al^{-\alpha}$ into Eq. (3) results in the Paretian survival probability $\mathbf{P}_{>}(x) = (x/l)^{-\alpha}$ ($x \geqslant l$). Thus, the Paretian distributions arising from the Probabilistic interpretation of Pareto's discovery are, in fact, resolution-contingent *projections* of the power-law Poissonian tail function arising from the Poissonian interpretation of Pareto's discovery.

3. Fractal Poissonian populations

In the previous section we have seen that infinite Poissonian populations are multi-scale objects which cannot be represented by a single *generic member*.

Are there Poissonian populations which, nonetheless, can be described by a *single probability law*? If existing, such a Poissonian population must be intrinsically *scale-invariant*, or *fractal*: its generic members—after "naturally scaling" them with respect to their levels—should be governed by a common, level-free, probability law.

Let $\mathscr{P}$ be an infinite Poissonian population defined on the range $\mathscr{R}$, and let $\{\psi_l(\cdot)\}_{l\in\mathscr{R}}$ be a family of monotone functions—the "natural" scaling functions—mapping, respectively, the sub-ranges (l, U) onto a *common scale* $\mathscr{S}$. Applying the scaling functions to the population's generic members $X_1(l)$ ($l \in \mathscr{R}$) yields the population's *scaled generic members*

$$Y_1(l) = \psi_l(X_1(l)) \quad (l \in \mathscr{R}). \tag{4}$$

The rigorous definition of "Poissonian fractality" is as follows:

Definition 2. An infinite Poissonian population $\mathscr{P}$ is fractal if its scaled generic members $Y_1(l)$ ($l \in \mathscr{R}$)—taking values on the common scale $\mathscr{S}$—are governed by a common probability law (independent of the level l).

Combining Eqs. (3) and (4) together, we obtain the following "functional fractality equations":
If the scale functions $\{\psi_l(\cdot)\}_{l\in\mathscr{R}}$ are monotone increasing then

$$\mathbf{P}_>(y) = \frac{R(\psi_l^{-1}(y))}{R(l)} \quad (y \in \mathscr{S};\ l \in \mathscr{R}), \tag{5}$$

where $\mathbf{P}_>(\cdot)$ is the common survival probability of the scaled generic members $Y_1(l)$ ($l \in \mathscr{R}$).
If the scale functions $\{\psi_l(\cdot)\}_{l\in\mathscr{R}}$ are monotone decreasing then

$$\mathbf{P}_\leqslant(y) = \frac{R(\psi_l^{-1}(y))}{R(l)} \quad (y \in \mathscr{S}; l \in \mathscr{R}), \tag{6}$$

where $\mathbf{P}_\leqslant(\cdot)$ is the common cumulative distribution function of the scaled generic members $Y_1(l)$ ($l \in \mathscr{R}$).
We now turn to characterize the different classes of Fractal Poissonian Populations. The proofs of the assertions made in this Section are given in Appendix A.

3.1. Unbounded populations

Consider the case of unbounded populations—both the lower bound and the upper bound being infinite ($L = -\infty$ and $U = \infty$). The range $\mathscr{R}$ is thus the entire real line $(-\infty, \infty)$.
The "natural scaling" in this case measures the distance from the level; i.e., it is *additive*:

$$\psi_l(x) = x - l \quad (x \geqslant l). \tag{7}$$

The common scale $\mathscr{S}$ is the non-negative half-line $[0, \infty)$.
Solving Eq. (5) with respect to the scaling functions of Eq. (7) yields an exponential Poissonian tail function of the form

$$R(l) = a\exp\{-\alpha l\} \quad (l \text{ real}), \tag{8}$$

where a and α are arbitrary positive parameters. The common probability law is *Exponential* with PDF

$$f(y) = \alpha\exp\{-\alpha y\} \quad (y \geqslant 0). \tag{9}$$

Poissonian populations ranging over the entire real line shall henceforth be referred to as *real Poissonian populations*. Real Poissonian populations governed by the exponential Poissonian tail function of Eq. (8) shall henceforth be referred to as *Real Poissonian Fractals*.

3.2. Populations bounded from below

Consider the case of populations bounded from below—the lower bound being finite ($L > -\infty$), and the upper bound being infinite ($U = \infty$). The range $\mathscr{R}$ is thus the ray (L, ∞).
A possible scaling is the additive scaling of Eq. (7) (the common scale being the non-negative half-line). This scaling, however, has two drawbacks: first, it does not take into account the lower bound L; second, it yields the exponential Poissonian tail function of Eq. (8) which—in the case of a finite lower bound—corresponds to *finite* Poissonian populations.
The "natural scaling" in this case measures the distance from the lower bound, with respect to the distance of the level from the lower bound; i.e., it is *linear*:

$$\psi_l(x) = \frac{x - L}{l - L} \quad (x \geqslant l). \tag{10}$$

The common scale $\mathscr{S}$ is the ray $[1, \infty)$.

Solving Eq. (5) with respect to the scaling functions of Eq. (10) yields a power-law Poissonian tail function of the form

$$R(l) = \frac{a}{(l-L)^{\alpha}} \quad (l > L), \tag{11}$$

where a and α are arbitrary positive parameters. The common probability law is *Paretian* with PDF

$$f(y) = \frac{\alpha}{y^{1+\alpha}} \quad (y \geqslant 1). \tag{12}$$

With no loss of generality, the lower bound can be set to be zero. Poissonian populations ranging over the positive half-line shall henceforth be referred to as *positive Poissonian populations*. Positive Poissonian populations governed by the power-law Poissonian tail function of Eq. (11) (with lower bound $L=0$) shall henceforth be referred to as *Positive Poissonian Fractals*.

3.3. Populations bounded from above

Consider the case of populations bounded from above—the lower bound being infinite ($L=-\infty$), and the upper bound being finite ($U<\infty$). The range $\mathcal{R}$ is thus the ray $(-\infty, U)$.

The "natural scaling" in this case—analogous to the case of populations bounded from below—measures the distance from the upper bound, with respect to the distance of the level from the upper bound; i.e., it is *linear*:

$$\psi_l(x) = \frac{U-x}{U-l} \quad (l \leqslant x < U). \tag{13}$$

The common scale $\mathcal{S}$ is the unit interval $(0, 1]$.

Solving Eq. (6) with respect to the scaling functions of Eq. (13) yields a power-law Poissonian tail function of the form

$$R(l) = a(U-l)^{\alpha} \quad (l < U), \tag{14}$$

where a and α are arbitrary positive parameters. The common probability law is *Beta* with PDF

$$f(y) = \alpha y^{\alpha-1} \quad (0 < y \leqslant 1). \tag{15}$$

With no loss of generality, the upper bound can be set to be zero. Poissonian populations ranging over the negative half-line shall henceforth be referred to as *negative Poissonian populations*. Negative Poissonian populations governed by the power-law Poissonian tail function of Eq. (14) (with upper bound $U=0$) shall henceforth be referred to as *Negative Poissonian Fractals*.

3.4. Bounded populations

Consider the case of bounded populations—both the lower bound and the upper bound being finite ($L > -\infty$ and $U < \infty$). The range $\mathcal{R}$ is thus the finite interval (L, U).

A possible scaling is the linear scaling of Eq. (13) (the common scale being the unit interval $(0, 1]$). This scaling, however, has two drawbacks: first, it does not take into account the lower bound L; second, it yields the power-law Poissonian tail function of Eq. (14) which—in the case of a finite lower bound—corresponds to *finite* Poissonian populations.

The linear scaling of Eq. (10) is non-admissible since it fails to map the sub-ranges (l, U) onto a common scale. Indeed, the sub-range (l, U) is mapped onto the interval $[1, (U-L)/(l-L))$—which is contingent on level l. However, taking the *logarithm* of the linear scaling of Eq. (10), and *normalizing* it, produces an admissible scaling. Namely, we obtain the following *logarithmic* "natural scaling":

$$\psi_l(x) = \frac{\ln\left(\dfrac{x-L}{l-L}\right)}{\ln\left(\dfrac{U-L}{l-L}\right)} \quad (l \leqslant x < U). \tag{16}$$

The common scale $\mathcal{S}$ is the unit interval $[0, 1)$.

Solving Eq. (5) with respect to the scaling functions of Eq. (16) yields a logarithmic Poissonian tail function of the form

$$R(l) = a\left(-\ln\left(\frac{l-L}{U-L}\right)\right)^{\alpha} \quad (U<l<L), \tag{17}$$

where a and α are arbitrary positive parameters. The common probability law is *Beta* with PDF

$$f(y) = \alpha(1-y)^{\alpha-1} \quad (0 \leqslant y < 1). \tag{18}$$

With no loss of generality, the range can be set to be the unit interval $(0, 1)$. Poissonian populations ranging over the unit interval shall henceforth be referred to as *unitary Poissonian populations*. Unitary Poissonian populations governed by the logarithmic Poissonian tail function of Eq. (17) (with lower bound $L = 0$ and upper bound $U = 1$) shall henceforth be referred to as *Unitary Poissonian Fractals*.

3.5. An interim summary

Let us pause and summarize the results obtained so far.

Four classes of Fractal Poissonian Populations were characterized—Real, Positive, Negative, and Unitary. The Poissonian tail functions of the four classes turned out to admit the functional form

$$R(l) = a(G(l))^{\alpha} \quad (l \in \mathscr{R}), \tag{19}$$

where a and α are arbitrary positive parameters, and where the function $G(\cdot)$ is an underlying parameter-free "generator":

(I) *Real Poissonian Fractals*—ranging over the real line, and governed by the exponential generator $G(l) = \exp\{-l\}$ (l real).

(II) *Positive Poissonian Fractals*—ranging over the positive half-line, and governed by the harmonic generator $G(l) = 1/l$ ($l>0$).

(III) *Negative Poissonian Fractals*—ranging over the negative half-line, and governed by the linear generator $G(l) = -l$ ($l<0$).

(IV) *Unitary Poissonian Fractals*—ranging over the unit interval, and governed by logarithmic generator $G(l) = -\ln(l)$ ($0<l<1$).

We note that it is possible to transform from one fractal class to another via a point-to-point mapping which maps each point x of the "input" Fractal Poissonian Population to a point $y = f(x)$ of the "output" Fractal Poissonian Population. The transformation-mappings between the four fractal classes are given in Table 1 below. The verification of these transforms is an immediate consequence of the "displacement theorem" of the theory of Poisson processes (see [21, Section 5.5]).

Table 1
Transformations between the four Fractal Poissonian Populations

	Real	Positive	Negative	Unitary
Real	x	$\exp\{x\}$	$-\exp\{-x\}$	$\exp\{-\exp\{-x\}\}$
Positive	$\ln(x)$	x	$-1/x$	$\exp\{-1/x\}$
Negative	$-\ln(-x)$	$-1/x$	x	$\exp\{x\}$
Unitary	$-\ln(-\ln(x))$	$-1/\ln(x)$	$\ln(x)$	x

In order to transform from the fractal class of row i to the fractal class of column j one has to apply the point-to-point mapping $y = f(x)$ appearing in cell (i,j).

4. Renormalization of Poissonian populations

In this section we explore the intimate connection between Fractal Poissonian Populations and the operation of renormalization. The proofs of the assertions made in this section are given Appendix A.

4.1. Renormalization

Let $\mathscr{P}_1, \ldots, \mathscr{P}_k$ be IID copies of a Poissonian population $\mathscr{P}$, defined on the range $\mathscr{R}$, and governed by the Poissonian tail function $R(\cdot)$. Consider the *renormalization* of the k populations, conducted by the following two-step procedure: (i) the populations are combined together to form their k-union $\mathscr{P}_1 \cup \cdots \cup \mathscr{P}_k$; and; (ii) each member of the k-union is re-scaled by a renormalization function $\phi_k(\cdot)$—a monotone-increasing map from the range $\mathscr{R}$ onto itself.

The resulting renormalized population, defined on the range $\mathscr{R}$, is given by

$$\mathscr{P}^{(k)} := \{\phi_k(x)\}_{x \in \mathscr{P}_1 \cup \cdots \cup \mathscr{P}_k}. \tag{20}$$

An immediate consequence of the ‘‘displacement theorem’’ of the theory of Poisson processes (see [21, Section 5.5]) is the following result:

Proposition 3. *The renormalized population $\mathscr{P}^{(k)}$ is a Poissonian population governed by the Poissonian tail function*

$$R^{(k)}(l) := kR(\phi_k^{-1}(l)) \quad (l \in \mathscr{R}). \tag{21}$$

Alternatively, one can take a Poissonian population with Poissonian tail function $kR(\cdot)$ (k being an arbitrary positive parameter), and re-scale each of its members by a renormalization function $\phi_k(\cdot)$ (a monotone-increasing map from the range $\mathscr{R}$ onto itself). The renormalized population is Poissonian with Poissonian tail function given by Eq. (21)—though now the scaling parameter is *continuous* ($k > 0$) rather than discrete ($k = 1, 2, \ldots$).

4.2. Renormalization consistency

For the renormalization to be *consistent*, it must be *commutative*: an mth-order renormalization, followed by a kth-order renormalization, must be identical to a (km)th-order renormalization (k and m being either discrete or continuous). Applied to the renormalized Poissonian tail functions (given by Eq. (21)), commutativity implies that $kmR(\phi_{km}^{-1}(l)) = kmR(\phi_m^{-1}(\phi_k^{-1}(l)))$. This, in turn (after a bit of algebra), implies that the renormalization functions must satisfy the ‘‘consistency condition’’:

$$\phi_{km}(x) = \phi_k(\phi_m(x)) \tag{22}$$

(k and m being either discrete or continuous). It is straightforward to observe that (β being an arbitrary real parameter):

A set $\{\phi_k(\cdot)\}_k$ of *translational* renormalization functions—defined on the real line—is consistent if and only if it is of the form

$$\phi_k(x) = \beta \ln(k) + x. \tag{23}$$

A set $\{\phi_k(\cdot)\}_k$ of *multiplicative* renormalization functions—defined on either the real line, the positive half-line, or the negative half-line—is consistent if and only if it is of the form

$$\phi_k(x) = k^\beta x. \tag{24}$$

A set $\{\phi_k(\cdot)\}_k$ of *power-law* renormalization functions—defined on either the positive half-line or the unit interval—is consistent if and only if it is of the form

$$\phi_k(x) = x^{k^\beta}. \tag{25}$$

4.3. Renormalization fixed points I

A Poissonian population $\mathscr{P}$ is a *fixed point* of the renormalization if the operation of renormalization leaves it *statistically unchanged*. Namely: The renormalized Poissonian population $\mathscr{P}^{(k)}$ equals, *in law*, the initial Poissonian population $\mathscr{P}$. This holds if and only if the Poissonian tail function $R^{(k)}(\cdot)$ of the renormalized population $\mathscr{P}^{(k)}$ equals the Poissonian tail function $R(\cdot)$ of the initial population $\mathscr{P}$. Using Eq. (21), we conclude that:

Proposition 4. *A Poissonian population $\mathscr{P}$ is a fixed point of the renormalization if and only if its Poissonian tail function $R(\cdot)$ satisfies the "functional renormalization equation":*

$$R(\phi_k(x)) = kR(x) \quad (x \in \mathscr{R}) \tag{26}$$

for all scaling functions $\phi_k(\cdot)$.

The four following cases—implied by Proposition 4—are of particular interest (a and α being arbitrary positive parameters):

(I) *Translational renormalization on the real line* $(-\infty, \infty)$. Solving Eq. (26) with respect to the translational renormalization functions of Eq. (23) yields the exponential Poissonian tail function

$$R(l) = a\exp\{-\alpha l\} \quad (l \text{ real}). \tag{27}$$

Hence, we conclude that: *a real Poissonian population is fractal if and only if it is a fixed point of a translational renormalization.*

(II) *Multiplicative renormalization on the positive half-line* $(0, \infty)$. Solving Eq. (26) with respect to the multiplicative renormalization functions of Eq. (24) yields the power-law Poissonian tail function

$$R(l) = al^{-\alpha} \quad (l > 0). \tag{28}$$

Hence, we conclude that: *a positive Poissonian population is fractal if and only if it is a fixed point of a multiplicative renormalization.*

(III) *Multiplicative renormalization on the negative half-line* $(-\infty, 0)$. Solving Eq. (26) with respect to the multiplicative renormalization functions of Eq. (24) yields the power-law Poissonian tail function

$$R(l) = a(-l)^{\alpha} \quad (l < 0). \tag{29}$$

Hence, we conclude that: *a negative Poissonian population is fractal if and only if it is a fixed point of a multiplicative renormalization.*

(IV) *Power-law renormalization on the unit interval* $(0, 1)$. Solving Eq. (26) with respect to the power-law renormalization functions of Eq. (25) yields the logarithmic Poissonian tail function

$$R(l) = a(-\ln(l))^{\alpha} \quad (0 < l < 1). \tag{30}$$

Hence, we conclude that: *a unitary Poissonian population is fractal if and only if it is a fixed point of a power-law renormalization.*

4.4. Renormalization fixed points II

In the previous subsection we sought the fixed-points of *given* renormalizations. However, the question can be turned around: Given a Poissonian population $\mathscr{P}$, is there a renormalization of which $\mathscr{P}$ is its fixed point? The answer turns out to be affirmative. Indeed, Eq. (26) implies that:

Proposition 5. *A Poissonian population $\mathscr{P}$, governed by the Poissonian tail function $R(\cdot)$, is a renormalization fixed point if and only if the renormalization functions $\{\phi_k(\cdot)\}_k$ are given by*

$$\phi_k(x) = R^{-1}(kR(x)) \quad (x \in \mathscr{R}). \tag{31}$$

Note that renormalization functions admitting the form of Eq. (31) automatically satisfy the "consistency condition" of Eq. (22).

Applying Proposition 5 to the four classes of Fractal Poissonian Populations obtained in Section 3 yields back translational, multiplicative, and power-law renormalizations (a and α being arbitrary positive parameters):

(I) *Real Poissonian Fractals* ($R(l) = a\exp\{-\alpha l\}$; l real):

$$\phi_k(x) = x - \frac{1}{\alpha}\ln(k) \quad (x \text{ real}). \tag{32}$$

(II) *Positive Poissonian Fractals* ($R(l) = al^{-\alpha}$; $l>0$):

$$\phi_k(x) = k^{-1/\alpha}x \quad (x>0). \tag{33}$$

(III) *Negative Poissonian Fractals* ($R(l) = a(-l)^{\alpha}$; $l<0$):

$$\phi_k(x) = k^{1/\alpha}x \quad (x<0). \tag{34}$$

(IV) *Unitary Poissonian Fractals* ($R(l) = a(-\ln(l))^{\alpha}$; $0<l<1$):

$$\phi_k(x) = x^{k^{1/\alpha}} \quad (0<x<1). \tag{35}$$

4.5. Basins of attraction

Consider a renormalization whose fixed point is the Poissonian population $\mathscr{P}_*$. What is the *basin of attraction* of the fixed point $\mathscr{P}_*$? That is: What are the Poissonian populations $\mathscr{P}$ whose renormalizations $\mathscr{P}^{(k)}$ converge, in the limit $k \to \infty$, *in law*, to the renormalization fixed point $\mathscr{P}_*$? The answer is given by the following result:

Proposition 6. *The basin of attraction of the renormalization's fixed point $\mathscr{P}_*$ is composed of all Poissonian populations $\mathscr{P}$ whose Poissonian tail functions $R(\cdot)$ satisfy the asymptotic condition*

$$\lim_{x\to U} \frac{R(x)}{R_*(x)} = 1, \tag{36}$$

where $R_*(\cdot)$ is $\mathscr{P}_*$'s Poissonian tail function.

The proof of Proposition 6 is given in Appendix A.

Proposition 6 implies that Poissonian populations, in the context of renormalization, are governed by the asymptotic behavior of their Poissonian tail functions at the upper bound. The range, however, must also be taken into account—as the following example demonstrates:

Consider a Poissonian population $\mathscr{P}$, with a finite upper bound, whose Poissonian tail function satisfies the power-law asymptotics $\lim_{x\to U} R(x)/F(x) = 1$, where $F(x) = a(U-x)^{\alpha}$ ($x<U$; a and α being arbitrary positive parameters).

If the lower bound is infinite then, with no loss of generality, the range can be taken to be the negative half-line. In this case Proposition 6 implies that the Poissonian population $\mathscr{P}$ belongs to the basin of attraction of the *Negative Poissonian Fractal* characterized by the Poissonian tail function $R_*(x) = a(-x)^{\alpha}$ ($x<0$).

If the lower bound is finite then, with no loss of generality, the range can be taken to be the unit interval. In this case Proposition 6 implies that the Poissonian population $\mathscr{P}$ belongs to the basin of attraction of the *Unitary Poissonian Fractal* characterized by the Poissonian tail function $R_*(x) = a(-\ln(x))^{\alpha}$ ($0<x<1$).

This example shows that the *same* upper-bound asymptotics can lead to *different* renormalization fixed points—depending on the *range* of the Poissonian population under consideration.

5. Maximum projections

In this section we study the maxima of Poissonian populations.

5.1. Maxima of Poissonian populations

Let $M_{\mathscr{P}}$ denote the *maximum* of a Poissonian population $\mathscr{P}$, defined on the range $\mathscr{R}$, and governed by the Poissonian tail function $R(\cdot)$. Namely, set $M_{\mathscr{P}} := \max_{x \in \mathscr{P}}\{x\}$ to be the maximum value amongst the values of all the population members.

Clearly, the maximum $M_{\mathscr{P}}$ is less or equal to the level l if and only if the Poissonian population $\mathscr{P}$ has no points above the level l ($l \in \mathscr{R}$). That is: $\{M_{\mathscr{P}} \leqslant l\} = \{\mathscr{P}(l) = \emptyset\} = \{N(l) = 0\}$ (recall that $\mathscr{P}(l)$ denotes the sub-population of points above the level l, and that $N(l)$ denotes the size of this sub-population).

Since the random variable $N(l)$ is Poisson-distributed with mean $R(l)$ (recall Eq. (1)), we obtain that the cumulative distribution function of the maximum $M_{\mathscr{P}}$ is given by

$$\mathbf{P}_{\leqslant}(l) = \exp\{-R(l)\} \quad (l \in \mathscr{R}). \tag{37}$$

Eq. (37) implies that there is a one-to-one correspondence between the maximum-distribution and the Poissonian tail function $R(\cdot)$. Since the Poissonian tail function $R(\cdot)$ characterizes the Poissonian population $\mathscr{P}$, we obtain that there is a statistical one-to-one correspondence between Poissonian populations and their associated maxima.

5.2. Maxima of fractal poissonian populations

Exploiting the one-to-one correspondence between Poissonian populations and their maxima, we conclude that (a and α being arbitrary positive parameters):

(I) *Real Poissonian Fractals*: a real Poissonian population is fractal if and only if its maximum is *Gumbel-distributed*—with cumulative distribution function of the form

$$\mathbf{P}_{\leqslant}(l) = \exp\{-a\exp\{-\alpha l\}\} \quad (l \text{ real}). \tag{38}$$

(II) *Positive Poissonian Fractals*: a positive Poissonian population is fractal if and only if its maximum is *Fréchet-distributed*—with cumulative distribution function of the form

$$\mathbf{P}_{\leqslant}(l) = \exp\{-al^{-\alpha}\} \quad (l > 0). \tag{39}$$

(III) *Negative Poissonian Fractals*: a negative Poissonian population is fractal if and only if its maximum is *Weibull-distributed*—with cumulative distribution function of the form

$$\mathbf{P}_{\leqslant}(l) = \exp\{-a(-l)^{\alpha}\} \quad (l < 0). \tag{40}$$

(IV) *Unitary Poissonian Fractals*: a unitary Poissonian population is fractal if and only if its maximum is governed by a cumulative distribution function of the form

$$\mathbf{P}_{\leqslant}(l) = \exp\{-a(-\ln(l))^{\alpha}\} \quad (0 < l < 1). \tag{41}$$

5.3. Extreme value distributions

Eqs. (38)–(40) imply an intimate connection between Real, Positive, and Negative Poissonian Fractals and Extreme Value Theory.

Given a sequence $\{\xi_n\}_{n=1}^{\infty}$ of IID random variables, *Extreme Value Theory* studies the asymptotic behavior of the linearly scaled maxima of the ξs. Namely: the limiting probability law of

$$\hat{M}_n := \frac{\max\{\xi_1, \ldots, \xi_n\} - b_n}{a_n} \quad (\text{as } n \to \infty), \tag{42}$$

where $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are properly chosen scaling coefficients.

Extreme Value Theory is of major importance in the analysis of rare and "catastrophic" events such as floods in hydrology, large claims in insurance, crashes in finance, material failure in corrosion analysis, etc. For both the theory and applications of Extreme Value Theory the readers are referred to [23,24].

The "Central Limit Theorem" of Extreme Value Theory asserts that the linearly scaled maxima have three possible (non-degenerate) limiting probability laws—referred to as the *Extreme Value* distributions: *Gumbel*—supported on the entire real line; *Fréchet*—supported on the positive half-line; and, *Weibull*—supported on the negative half-line.

Thus, the three Extreme Value distributions—Gumbel, Fréchet, and Weibull—correspond, respectively, to Real, Positive, and Negative Poissonian Fractals.

The reason why there is no Extreme Value distribution corresponding to Unitary Poissonian Fractals is *nonlinearity*: the scaling of Extreme Value Theory is *linear* (Eq. (42))—whereas the scaling and the renormalization of Unitary Poissonian Fractals is *nonlinear* (Eqs. (16) and (35)). The intrinsic nonlinearity of Unitary Poissonian Fractals renders their "maximum projections" beyond the realm of Extreme Value distributions.

6. Sum projections

In this section we study the sums of Poissonian populations. Infinite random populations with non-zero lower bounds ($L \neq 0$) are non-summable. In this section we therefore consider only Poissonian populations with zero lower bounds ($L = 0$)—i.e., positive Poissonian populations.

6.1. Sums of positive Poissonian populations

Let $S_{\mathscr{P}}$ denote the *sum* of a Poissonian population $\mathscr{P}$, defined on the range $\mathscr{R} = (0, U)$ ($U \leqslant \infty$), and governed by the Poissonian tail function $R(\cdot)$. Namely, set $S_{\mathscr{P}} := \sum_{x \in \mathscr{P}} x$ to be the sum of all the population members' values. For example, if the value measured is wealth then $S_{\mathscr{P}}$ is the cumulative wealth of the entire population.

Campbell's theorem of the theory of Poisson processes (see [21, Section 3.2]) implies that the Poissonian population $\mathscr{P}$ is *summable*—i.e., that its sum $S_{\mathscr{P}}$ is convergent—if and only if the population's Poissonian tail function $R(\cdot)$ is integrable at the origin: $\int_0^1 R(x)\,\mathrm{d}x < \infty$. Campbell's theorem further implies that if the Poissonian population is summable then the *Laplace transform* of the sum $S_{\mathscr{P}}$ is given by

$$\mathscr{L}(\theta) = \exp\left\{-\theta \int_0^{\infty} \exp\{-\theta x\} R(x)\,\mathrm{d}x\right\} \quad (\theta \geqslant 0). \tag{43}$$

Eq. (43) implies that there is a one-to-one correspondence between the sum-distribution and the Poissonian tail function $R(\cdot)$ (via its Laplace transform $\widetilde{R}(\theta) = \int_0^{\infty} \exp\{-\theta x\} R(x)\,\mathrm{d}x \quad (\theta \geqslant 0)$). Since the Poissonian tail function $R(\cdot)$ characterizes the Poissonian population $\mathscr{P}$, we obtain that there is a statistical one-to-one correspondence between summable Poissonian populations and their associated sums.

The *cumulants* of the sum $S_{\mathscr{P}}$, if convergent, are given by

$$\mathscr{C}(m) = m \int_0^{\infty} x^{m-1} R(x)\,\mathrm{d}x \quad (m = 1, 2, \ldots). \tag{44}$$

6.2. Sums of Fractal Poissonian Populations

Exploiting the one-to-one correspondence between summable Poissonian populations and their sums, we obtain that (a and α being arbitrary positive parameters):

(II) *Positive Poissonian Fractals*: a summable positive Poissonian population is fractal if and only if its sum is *Lévy Stable*—with Laplace transform of the form

$$\mathscr{L}(\theta) = \exp\{-a\Gamma(1-\alpha)\theta^{\alpha}\} \quad (\theta \geqslant 0), \tag{45}$$

where the exponent is in the range $0 < \alpha < 1$.

(IV) *Unitary Poissonian Fractals*: a summable unitary Poissonian population is fractal if and only if its sum is governed by the sequence of cumulants

$$\mathscr{C}(m) = \frac{a\Gamma(1+\alpha)}{m^{\alpha}} \quad (m = 1, 2, \ldots). \tag{46}$$

Note the power-law structure of the Log-Laplace transform of the sums of summable Positive Poissonian Fractals (Eq. (45)), and the power-law structure of the cumulants of the sums of Unitary Poissonian Fractals (Eq. (46)).

6.3. Lévy Stable distributions

Eq. (45) implies an intimate connection between Positive Poissonian Fractals and Lévy Stable distributions.

Given a sequence $\{\xi_n\}_{n=1}^{\infty}$ of IID random variables, Fluctuation Theory studies the asymptotic behavior of the linearly scaled sums of the ξ's. Namely: the limiting probability law of

$$\hat{S}_n := \frac{\{\xi_1 + \cdots + \xi_n\} - b_n}{a_n} \quad (\text{as } n \to \infty), \tag{47}$$

where $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are properly chosen scaling coefficients.

The generalized Central Limit Theorem—due to Lévy, Gnedenko, and Kolmogorov [25,26]—asserts that the only possible (non-degenerate) limiting probability laws of the linearly scaled sums are the *Lévy Stable* distributions. In the case of positive valued random variables, the Lévy Stable distributions are characterized by Laplace transforms of the form $\exp\{-a\Gamma(1-\alpha)\theta^{\alpha}\}$ ($\theta \geqslant 0$; a being an arbitrary positive coefficient and the exponent taking values in the range $0 < \alpha < 1$).

Thus, the Lévy Stable distributions correspond to summable Positive Poissonian Fractals.

As in the case of Extreme Value distributions, the reason why there is no Lévy Stable distribution corresponding to Unitary Poissonian Fractals is *nonlinearity*: the scaling of the Central Limit Theorem is *linear* (Eq. (47))—whereas the scaling and the renormalization of Unitary Poissonian Fractals is *nonlinear* (Eqs. (16) and (35)). The intrinsic nonlinearity of Unitary Poissonian Fractals renders their "sum projections" beyond the realm of Lévy Stable distributions.

7. Conclusions

Random populations represented by Poisson-scattered collections of real-valued points were considered. A Poissonian population can be either finite or infinite. If finite then the population is a single-scale object representable by a "generic member". If infinite, then the population is a multi-scale object which no single "generic member" can represent.

For infinite Poissonian populations we defined fractality via the elemental notion of scale-invariance, and characterized four classes of fractals:

(I) *Real Poissonian Fractals*: ranging over the real line, and governed by exponential Poissonian tail functions. Real Poissonian Fractals are the fixed points of translational renormalizations on the real line, and are in one-to-one correspondence with the Gumbel-class of Extreme Value distributions.

Table 2
The characteristic features of the four classes of Fractal Poissonian Populations (a and α are arbitrary positive parameters)

	Real	Positive	Negative	Unitary
Range $\mathscr{R}$	$(-\infty, \infty)$	$(0, \infty)$	$(-\infty, 0)$	$(0, 1)$
Tail $R(l) = (l \in \mathscr{R})$	$a \exp\{-\alpha l\}$	$a l^{-\alpha}$	$a(-l)^{\alpha}$	$a(-\ln(l))^{\alpha}$
Scale $\mathscr{S}$	$[0, \infty)$	$[1, \infty)$	$(0, 1]$	$[0, 1)$
Scaling $\psi_l(x) = (l \leqslant x \in \mathscr{R})$	$x - l$	x/l	x/l	$\dfrac{\ln(x/l)}{\ln(1/l)}$
Common PDF $f(y) = (y \in \mathscr{S})$	$\alpha \exp\{-\alpha y\}$	$\alpha y^{-\alpha-1}$	$\alpha y^{\alpha-1}$	$\alpha(1-y)^{\alpha-1}$
Renormalization $\phi_k(x) = (x \in \mathscr{R})$	$x - \dfrac{1}{\alpha}\ln(k)$	$k^{-1/\alpha}x$	$k^{1/\alpha}x$	$x^{k^{1/\alpha}}$
EV maxima	Gumbel	Fréchet	Weibull	—

(II) *Positive Poissonian Fractals*: ranging over the positive half-line, and governed by power-law Poissonian tail functions with negative exponents. Positive Poissonian Fractals are the fixed points of multiplicative renormalizations on the positive half-line; are in one-to-one correspondence with the Fréchet-class of Extreme Value distributions; and, if summable, are in one-to-one correspondence with the class of positive-valued Lévy Stable distributions.

(III) *Negative Poissonian Fractals*: ranging over the negative half-line, and governed by power-law Poissonian tail functions with positive exponents. Negative Poissonian Fractals are the fixed points of multiplicative renormalizations on the negative half-line, and are in one-to-one correspondence with the Weibull-class of Extreme Value distributions.

(IV) *Unitary Poissonian Fractals*: ranging over the unit interval, and governed by logarithmic Poissonian tail functions. Unitary Poissonian Fractals are the fixed points of power-law renormalizations on the unit interval, and are in one-to-one correspondence with the class of positive-valued distributions characterized by power-law cumulants.

Table 2 summarizes the characteristic features of the four classes of Fractal Poissonian Populations.

Real, Positive, and Negative Poissonian Fractals are "linear fractals"—their associated scaling functions and renormalization functions being linear. Unitary Poissonian Fractals, on the other hand, are "nonlinear fractals"—their associated scaling functions and renormalization functions being nonlinear. This intrinsic nonlinearity of Unitary Poissonian Fractals renders their maximum and sum projections, respectively, beyond the realm of Extreme Value distributions and Lévy Stable distributions.

Appendix A

A.1. Characterization of Fractal Poissonian Populations

In this subsection of Appendix A we prove the characterization results of Fractal Poissonian Populations asserted in Section 3.

A.1.1. Unbounded populations

Set $F(y) = \mathbf{P}_{>}(y)$ $(y \geqslant 0)$ and $a = R(0)$.

Noting that $\psi_l^{-1}(y) = l + y$, Eq. (5) gives

$$F(y) = \frac{R(l+y)}{R(l)} \quad (l \text{ real}; \; y \geqslant 0). \tag{48}$$

Eq. (48) implies that

$$F(y_1 + y_2) = \frac{R(l + y_1 + y_2)}{R(l)} = \frac{R((l + y_2) + y_1)}{R(l + y_2)} \frac{R(l + y_2)}{R(l)} = F(y_1)F(y_2) \quad (y_1, y_2 \geqslant 0). \tag{49}$$

Eq. (49) implies that the function $F(\cdot)$ is an exponential. Since $F(\cdot)$ is a survival probability, we obtain that

$$F(y) = \exp\{-\alpha y\} \quad (y \geqslant 0), \tag{50}$$

where α is a positive parameter.

Taking $l = 0$ in Eq. (48) gives

$$R(y) = R(0)F(y) = a\exp\{-\alpha y\} \quad (y \geqslant 0). \tag{51}$$

Taking $l = -y$ in Eq. (48) gives

$$R(-y) = \frac{R(0)}{F(y)} = a\exp\{\alpha y\} \quad (y \geqslant 0). \tag{52}$$

Combining Eqs. (51) and (52) together, we conclude that

$$R(l) = a\exp\{-\alpha l\} \quad (l \text{ real}). \tag{53}$$

A.1.2. Populations bounded from below

Set $\theta = (l - L)$ $(\theta > 0)$; $F(y) = \mathbf{P}_{>}(y)$ $(y \geqslant 1)$; $G(t) = R(L + t)$ $(t > 0)$; and, $a = G(1) = R(L + 1)$.

Noting that $\psi_l^{-1}(y) = L + \theta y$, Eq. (5) gives

$$F(y) = \frac{G(\theta y)}{G(\theta)} \quad (\theta > 0; y \geqslant 1). \tag{54}$$

Eq. (54) implies that

$$F(y_1 y_2) = \frac{G(\theta y_1 y_2)}{G(\theta)} = \frac{G((\theta y_2)y_1)}{G(\theta y_2)}\frac{G(\theta y_2)}{G(\theta)} = F(y_1)F(y_2) \quad (y_1, y_2 \geqslant 1). \tag{55}$$

Eq. (55) implies that the function $F(\cdot)$ is a power-law. Since $F(\cdot)$ is a survival probability, we obtain that

$$F(y) = y^{-\alpha} \quad (y \geqslant 1), \tag{56}$$

where α is a positive parameter.

Taking $l = L + 1$ in Eq. (54) gives

$$G(y) = G(1)F(y) = ay^{-\alpha} \quad (y \geqslant 1). \tag{57}$$

Eq. (57), in turn, implies that

$$R(l) = G(l - L) = a(l - L)^{-\alpha} \quad (l > L). \tag{58}$$

A.1.3. Populations bounded from above

Set $\theta = (U - l)$ $(\theta > 0)$; $F(y) = \mathbf{P}_{\leqslant}(y)$ $(0 < y \leqslant 1$; $G(t) = R(U - t)$ $(t > 0)$; and, $a = G(1) = R(U - 1)$.

Noting that $\psi_l^{-1}(y) = U - \theta y$, Eq. (6) gives

$$F(y) = \frac{G(\theta y)}{G(\theta)} \quad (\theta > 0; 0 < y \leqslant 1). \tag{59}$$

Eq. (59) implies that

$$F(y_1 y_2) = \frac{G(\theta y_1 y_2)}{G(\theta)} = \frac{G((\theta y_2)y_1)}{G(\theta y_2)}\frac{G(\theta y_2)}{G(\theta)} = F(y_1)F(y_2) \quad (0 < y_1, y_2 \leqslant 1). \tag{60}$$

Eq. (60) implies that the function $F(\cdot)$ is a power-law. Since $F(\cdot)$ is a cumulative distribution function, we obtain that

$$F(y) = y^{\alpha} \quad (0 < y \leqslant 1), \tag{61}$$

where α is a positive parameter.

Taking $l = U - 1$ in Eq. (59) gives

$$G(y) = G(1)F(y) = ay^{\alpha} \quad (0 < y \leqslant 1). \tag{62}$$

Eq. (62), in turn, implies that

$$R(l) = G(U - l) = a(U - l)^{\alpha} \quad (l < U). \tag{63}$$

A.1.4. Bounded populations

Set $\theta = (l - L)/(U - L)$ $(0 < \theta < 1)$; $F(y) = \mathbf{P}_{>}(y)$ $(0 < y < 1)$; $H(y) = F(1 - y)$ $(0 < y < 1)$; $G(t) = R(L + (U - L)t)$ $(0 < t < 1)$; and, $a = G(e^{-1})$.

Noting that $\psi_l^{-1}(y) = L + (U - L)\theta^{1-y}$, Eq. (5) gives

$$F(y) = \frac{R(L + (U - L)\theta^{1-y})}{R(L + (U - L)\theta)} = \frac{G(\theta^{1-y})}{G(\theta)} \;\Rightarrow\; H(y) = \frac{G(\theta^{y})}{G(\theta)} \quad (0 < \theta, y < 1). \tag{64}$$

Eq. (64) implies that

$$H(y_1 y_2) = \frac{G(\theta^{y_1 y_2})}{G(\theta)} = \frac{G((\theta^{y_2})^{y_1})}{G(\theta^{y_2})}\frac{G(\theta^{y_2})}{G(\theta)} = H(y_1)H(y_2) \quad (0 < y_1, y_2 \leqslant 1). \tag{65}$$

Eq. (65) implies that the function $H(\cdot)$ is a power-law. Since $F(\cdot)$ is a survival probability, we obtain that

$$H(y) = y^{\alpha} \;\Rightarrow\; F(y) = (1 - y)^{\alpha} \quad (0 < y < 1), \tag{66}$$

where α is a positive parameter.

Taking $\theta = e^{-1}$ in Eq. (64) gives

$$G(e^{-y}) = G(e^{-1})H(y) = ay^{\alpha} \;\Rightarrow\; G(t) = a(-\ln(t))^{\alpha} \quad (0 < t < 1). \tag{67}$$

Finally, Eq. (67) implies that

$$R(l) = a\left(-\ln\left(\frac{l - L}{U - L}\right)\right)^{\alpha} \quad (U < l < L). \tag{68}$$

A.2. Renormalization of Poissonian populations

In this subsection of Appendix A we prove the renormalization results asserted in Section 4. We consider the renormalization parameter k to be an arbitrary positive parameter $(k > 0)$.

A.2.1. Renormalization consistency

A set $\{\phi_k(\cdot)\}_k$ of *translational* renormalization functions is of the form $\phi_k(x) = f(k) + x$. Renormalization consistency (Eq. (22)) implies that the function $f(\cdot)$ must satisfy the condition $f(km) = f(k) + f(m)$—which, in turn, implies that $f(k) = \beta \ln(k)$, where β is an arbitrary real parameter.

A set $\{\phi_k(\cdot)\}_k$ of *multiplicative* renormalization functions is of the form $\phi_k(x) = f(k) \cdot x$. Renormalization consistency (Eq. (22)) implies that the function $f(\cdot)$ must satisfy the condition $f(km) = f(k) \cdot f(m)$—which, in turn, implies that $f(k) = k^{\beta}$, where β is an arbitrary real parameter.

A set $\{\phi_k(\cdot)\}_k$ of *power-law* renormalization functions is of the form $\phi_k(x) = x^{f(k)}$. Renormalization consistency (Eq. (22)) implies that the function $f(\cdot)$ must satisfy the condition $f(km) = f(k) \cdot f(m)$—which, in turn, implies that $f(k) = k^{\beta}$, where β is an arbitrary real parameter.

A.2.2. Translational renormalization on the real line

Substituting the translational renormalization functions of Eq. (23) into Eq. (26) yields the functional equation

$$R(\beta \ln(k) + x) = kR(x) \quad (x \text{ real};\; k > 0). \tag{69}$$

Setting $x=0$ and $\beta\ln(k)=y$ gives $R(y)=R(0)\exp\{y/\beta\}$ (y real). Since the Poissonian tail function is monotone decreasing from infinity to zero—it admits the form $R(y)=a\exp\{-\alpha y\}$ (y real), where a and α are arbitrary positive parameters.

A.2.3. Multiplicative renormalization on the positive half-line

Substituting the translational renormalization functions of Eq. (24) into Eq. (26) yields the functional equation

$$R(k^{\beta}x)=kR(x)\quad(x>0;k>0). \tag{70}$$

Setting $x=1$ and $k^{\beta}=y$ gives $R(y)=R(1)y^{1/\beta}$ ($y>0$). Since the Poissonian tail function is monotone decreasing from infinity to zero—it admits the form $R(y)=ay^{-\alpha}$ ($y>0$), where a and α are arbitrary positive parameters.

A.2.4. Multiplicative renormalization on the negative half-line

Substituting the translational renormalization functions of Eq. (24) into Eq. (26) yields the functional equation

$$R(k^{\beta}x)=kR(x)\quad(x<0;k>0). \tag{71}$$

Setting $x=-1$ and $k^{\beta}=y$ gives $R(-y)=R(-1)y^{1/\beta}$ ($y>0$). Since the Poissonian tail function is monotone decreasing from infinity to zero—it admits the form $R(y)=a(-y)^{\alpha}$ ($y<0$), where a and α are arbitrary positive parameters.

A.2.5. Power-law renormalization on the unit interval

Substituting the translational renormalization functions of Eq. (25) into Eq. (26) yields the functional equation

$$R(x^{k^{\beta}})=kR(x)\quad(0<x<1;k>0). \tag{72}$$

Setting $x=e^{-1}$ and $\exp\{-k^{\beta}\}=y$ gives $R(y)=R(e^{-1})(-\ln(y))^{1/\beta}$ ($0<y<1$). Since the Poissonian tail function is monotone decreasing from infinity to zero—it admits the form $R(y)=a(-\ln(y))^{\alpha}$ ($0<y<1$), where a and α are arbitrary positive parameters.

A.2.6. Proof of Proposition 6: basins of attraction

Let $R_*(\cdot)$ and $R(\cdot)$ denote, respectively, the Poissonian tail functions of the Poissonian populations $\mathscr{P}_*$ and $\mathscr{P}$.

Since the Poissonian population $\mathscr{P}_*$ is a renormalization fixed point, Proposition 5 implies that the renormalization functions $\{\phi_k(\cdot)\}_k$ are given by $\phi_k(\cdot)=R_*^{-1}(kR_*(\cdot))$. Hence, the inverses of the renormalization functions are given by $\phi_k^{-1}(\cdot)=R_*^{-1}((1/k)R_*(\cdot))$.

Note that the change of variables $x=\phi_k^{-1}(l)$—k and x being the variables—gives $k=R_*(l)/R_*(x)$. Also note that the limit $k\to\infty$ corresponds to the limit $x\to U$.

Proposition 3 implies that the Poissonian tail function $R^{(k)}(\cdot)$ of the renormalized population $\mathscr{P}^{(k)}$ is given by $R^{(k)}(\cdot)=kR(\phi_k^{-1}(\cdot))$. Hence, combining the above together, we obtain that

$$\lim_{k\to\infty}R^{(k)}(l)=\lim_{k\to\infty}kR(\phi_k^{-1}(l))=\lim_{x\to U}\frac{R_*(l)}{R_*(x)}R(x)=\left(\lim_{x\to U}\frac{R(x)}{R_*(x)}\right)\cdot R_*(l)\quad(l\in\mathscr{R}) \tag{73}$$

Eq. (73) implies that the Poissonian tail functions $R^{(k)}(\cdot)$ converge to the limit $R_*(\cdot)$ if and only if the asymptotic condition

$$\lim_{x\to U}\frac{R(x)}{R_*(x)}=1 \tag{74}$$

is satisfied. Since the renormalizations $\mathscr{P}^{(k)}$ converge, in the limit $k\to\infty$, *in law*, to the Poissonian population $\mathscr{P}_*$ if and only if their Poissonian tail function $R^{(k)}(\cdot)$ converge, in the limit $k\to\infty$, to the Poissonian tail function $R_*(\cdot)$—the proof is complete.

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