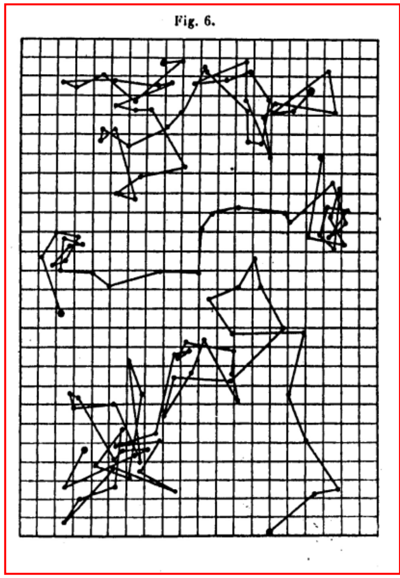
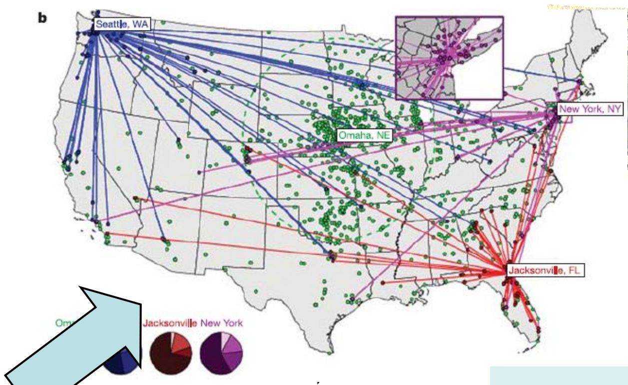


J.B. Perrin, 1909

"Normal" DE:

$$\frac{\partial}{\partial t} f(t,\vec{r}) = D\Delta f(t,\vec{r})$$



Fractional DE:

$$\frac{\partial^{\beta}}{\partial t^{\beta}} f(t,\vec{r}) = -D\left(-\Delta\right)^{\alpha/2} f(t,\vec{r})$$

Brockmann, Hufnagel, Geisel, 2006

Natural and Modified Forms of Distributed Order Fractional Diffusion Equations

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Review: A. Chechkin, I. Sokolov, J. Klafter, in: **Fractional Dynamics**,

S.C. Lim, R. Metzler, J. Klafter (Eds), World Scientific (2011 ?)

Outline

- Introduction: fractional kinetics, relation to random walk scheme, time and space fractional derivatives

- Time and space fractional diffusion equations in *normal* and *modified* forms, equivalence of the two forms

- Distributed order fractional derivative

- Distributed order fractional diffusion equations (DODE) in *normal* and *modified* forms

  - ✓ non-equivalence of the two forms

  - ✓ different regimes of anomalous diffusion

- Summary: the Table

Fractional kinetics: diffusion and kinetic equations with fractional derivatives⇔ "strange" kinetics (Shlesinger, Zaslavsky, Klafter, 1993): connected with deviations

	"Normal" Kinetics	Fractional Kinetics
Diffusion law	$\langle x^2(t) \rangle \propto t$	$\langle x^2(t) \rangle \propto t^{\mu} \quad , \quad \mu \neq 1$ $\langle x^2(t) \rangle \propto \ln^{\nu} t \quad , \quad \nu > 0$
Relaxation	Exponential	Non-exponential "Lévy flights in time"
Stationary state	Maxwell-Boltzmann equilibrium	Confined Lévy flights as non-Boltzmann stationarity

Simplest Types of "Fractionalisation"

- **time – fractional diff eq**

$$\frac{\partial}{\partial t} \to \frac{\partial^{\beta}}{\partial t^{\beta}} \quad , \quad 0 < \beta < 1$$

$$\frac{\partial f}{\partial t} = D \frac{\partial^2 f}{\partial x^2}$$

DIFFUSION EQ

Caputo/Riemann-Liouville **derivative**

- **space – fractional diff eq**

$$\frac{\partial^2}{\partial x^2} \to \frac{\partial^{\alpha}}{\partial |x|^{\alpha}} \quad , \quad 0 < \alpha < 2$$

Riesz derivative : symmetric combination of left/right side RL derivatives

- **multi-dimensional / anisotropic**
- **time-space / velocity fractional**

Question : ***Derivation ?***

Answer: ***from Generalized Master Equation*** **and/or**
and/or Generalized Langevin Description

Underlying physical picture: Random walk with long jumps and long waiting times

Random walk $x(t)$, PDF $f(x,t)$, consisting of

- **Random jumps :** $\xi_i = x(t_i) - x(t_{i-1})$ ⇨ Jump PDF : $\lambda(\xi)$ i=1,2,…
- **Random waiting times :** $\tau_i = t_i - t_{i-1}$ ⇨ Waiting time PDF : $w(\tau)$

Question : Diffusion equation for $f(x,t)$ in the long time – space limit ?

Answer : Depends on the asymptotic behavior of $\lambda(\xi)$ and $w(\tau)$.

- mean square displacement $\langle \xi^2 \rangle = \int_{-\infty}^{\infty} d\xi\, \xi^2 \lambda(\xi)$ either finite or infinite
- mean waiting time $\langle \tau \rangle = \int_0^{\infty} d\tau\, \tau\, w(\tau)$ either finite or infinite

Underlying physical picture: Random walk with long jumps and/or long waiting times

Belongs to domain of attraction of symmetric α-stable Lévy law

$\lambda(\xi) \propto 1/|\xi|^{1+\alpha}$

$0 < \alpha < 2, \langle \xi^2 \rangle = \infty$

$\langle \xi^2 \rangle$ *finite*

$\langle \tau \rangle$ *finite*

$w(\tau) \propto 1/\tau^{1+\beta}$

$0 < \beta < 1, \langle \tau \rangle = \infty$

Belongs to domain of attraction of one-sided α-stable Lévy law

Ordinary DE

$$\frac{\partial f}{\partial t} = D \frac{\partial^2 f}{\partial x^2}$$

Space fractional DE

$$\frac{\partial f}{\partial t} = D \frac{\partial^\alpha f}{\partial |x|^\alpha}$$

Time fractional DE

$$\frac{\partial^\beta f}{\partial t^\beta} = D \frac{\partial^2 f}{\partial x^2}$$

Space-time fractional DE

$$\frac{\partial^\beta f}{\partial t^\beta} = D \frac{\partial^\alpha f}{\partial |x|^\alpha}$$

Time fractional derivatives in Caputo and Riemann-Liouville forms

Fractional integral

$$J^{\beta} f(t) := \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} f(\tau) d\tau \quad , \quad \beta \in R^{+}$$

definition

Caputo

Riemann-Liouville

$$\frac{d^{\beta}}{dt^{\beta}} f(t) := J^{1-\beta} \frac{d}{dt} f(t)$$

$0 < \beta < 1$

$$D_t^{\beta} f(t) := \frac{d}{dt} J^{1-\beta} f(t)$$

$$\frac{d^{\beta}}{dt^{\beta}} f := \frac{1}{\Gamma(1-\beta)} \int_0^t d\tau \, (t-\tau)^{-\beta} \frac{d}{d\tau} f(\tau) \quad (1)$$

$$D_t^{\beta} f := \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t d\tau \frac{f(\tau)}{(t-\tau)^{\beta}} \quad (2)$$

Laplace transform

$$\frac{d^{\beta} f}{dt^{\beta}} \div s^{\beta} \tilde{f}(s) - s^{\beta-1} f(0)$$

$$D_t^{\beta} f \div s^{\beta} f(s)$$

Laplace (Fourier) transform pair

“natural” generalization (take $\beta = 1$)

Two Forms of Time Fractional Diffusion Equations

Caputo form

"normal" form

$$\frac{\partial^\beta}{\partial t^\beta} f(x,t) = K_\beta \frac{\partial^2}{\partial x^2} f(x,t)$$

$0 < \beta \le 1$

Riemann-Liouville form

"modified" form

$$\frac{\partial}{\partial t} f(x,t) = K_\beta D_t^{1-\beta} \frac{\partial^2}{\partial x^2} f(x,t)$$

$$f(x,t=0) = \delta(x)$$

Fourier-Laplace transform :

$$f(x,t) \quad \div \quad f(k,s) = \int_{-\infty}^{\infty} dx e^{ikx} \int_0^{\infty} dt e^{-st} f(x,t)$$

normal form ⇒ $$f(k,s) = \frac{s^{\beta-1}}{s^\beta + K_\beta k^2}$$ ⇐ **modified form**

Normal and modified forms are equivalent

SPACE FRACTIONAL DERIVATIVE VIA ITS FOURIER TRANSFORM

Fourier transform pair

$$f(x) \div f(k) = \int_{-\infty}^{\infty} dx\, e^{ikx} f(x) \quad , \quad f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-ikx} f(k)$$

1st derivative: $$\frac{df}{dx} \div -ikf(k)$$

2nd derivative: $$\frac{d^2 f}{dx^2} \div \left(-ik\right)^2 f(k) = -k^2 f(k)$$

Symmetric Riesz der.:

$$\frac{d^{\alpha} f}{d|x|^{\alpha}} \equiv -\left(-\Delta\right)^{\alpha/2} \div -|k|^{\alpha} f(k)$$

Coincide with the "usual" second order derivative :

$$\frac{d^2 f}{d|x|^2} = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left(-k^2\right) f(k) e^{-ikx} = \frac{d^2 f}{dx^2}$$

Two Forms of Space Fractional Diffusion Equations

"normal" form

$$\frac{\partial f}{\partial t} = K_\alpha \frac{\partial^\alpha f}{\partial |x|^\alpha}$$

$0 < \alpha \le 2$

"modified" form

$$\frac{\partial^{2-\alpha}}{\partial |x|^{2-\alpha}} \frac{\partial f}{\partial t} = -K_\alpha \frac{\partial^2}{\partial x^2} f$$

where

$$\frac{d^\alpha \phi(x)}{d |x|^\alpha} \div -|k|^\alpha \phi(k)$$

Characteristic function: $f(k,t) \div f(x,t)$

normal form $\Rightarrow$ $f(k,t) = \exp\left(-K_\alpha |k|^\alpha t\right)$ $\Leftarrow$ **modified form**

Normal and modified forms are equivalent

Applications of fractional diffusion / kinetic equations

Space fractional

- Fluid and plasma turbulence
- Strange diffusion on DNA
- Lévy flights of photons
- Propagation of light in fractal media
- Diffusion of guiding centers in turbulent magnetized plasmas
- Human travel
- Deterministic maps
- Hamiltonian chaos

Time fractional

- Transport in amorphous materials
- Transport of passive tracers in underground water
- Financial markets, stock prices
- Deterministic maps
- Hamiltonian chaos

See, e.g., R. Metzler, J. Klafter, *Phys Rep* 2000,
I. Sokolov, J. Klafter, A. Blumen, *Physics Today* 2002,
R. Metzler, A. Chechkin, J. Klafter, *Encyclopedia of Complexity and System Science*, 2009.

Normal and Anomalous diffusion

Invariance under the scale transformation

L: typical (characteristic) scale of the solution

Normal diffusion eq

$$\frac{\partial f}{\partial t} = D\frac{\partial^2 f}{\partial x^2}$$

$$t \to \lambda t\,,\ x \to \lambda^{1/2} x$$

$$L \propto t^{1/2}$$

$$\left\langle x^2(t) \right\rangle \propto t$$

Normal diffusion

Time-fractional eq

$$\frac{\partial^\beta}{\partial t^\beta} f(x,t) = D_\beta \frac{\partial^2}{\partial x^2} f(x,t)$$

$$t \to \lambda t\,,\ x \to \lambda^{\beta/2} x$$

$$L \propto t^{\beta/2}$$

$$\left\langle x^2(t) \right\rangle \propto t^\beta$$

Slow diffusion, $0 < \beta < 1$

Space fractional eq

$$\frac{\partial f}{\partial t} = D_\alpha \frac{\partial^\alpha f}{\partial |x|^\alpha}$$

$$t \to \lambda t,\ x \to \lambda^{1/\alpha} x$$

$$L \propto t^{1/\alpha}$$

$$\left\langle |x|^q \right\rangle^{2/q} \propto t^{2/\alpha}\,,\ q < \alpha$$

Fast diffusion, $0 < \alpha < 2$

- **However: many (most of ?) systems demonstrate non-scaling or multiscaling anomalous behavior e.g.,**
  - ✓ **crossover between different power laws,**
  - ✓ **non-power-law logarithmic behavior …**

Q: Is it possible to extend the notion of fractional derivative operator in order to describe such anomalous behavior ?

A: Different possibilities !

Possibility 1. Tempered α - stable Lévy distributions and exponentially truncated Lévy flights

Possibility 2. Diffusion equations with variable order derivatives

- **Time fractional, inhomoheneous media**

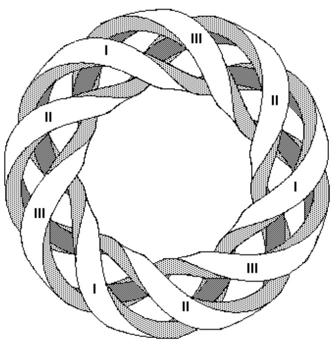
$$\frac{\partial f}{\partial t} = \frac{\partial^2}{\partial x^2} K(x) D_t^{1-\beta(x)} f$$

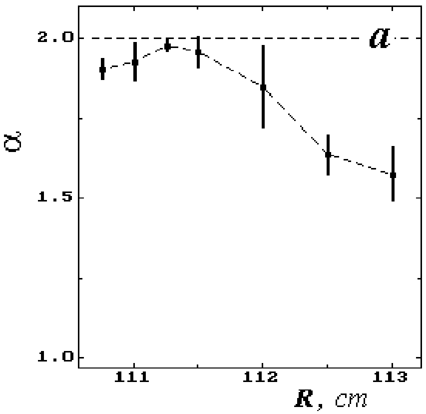
Ch, Gorenflo, Sokolov, 2005

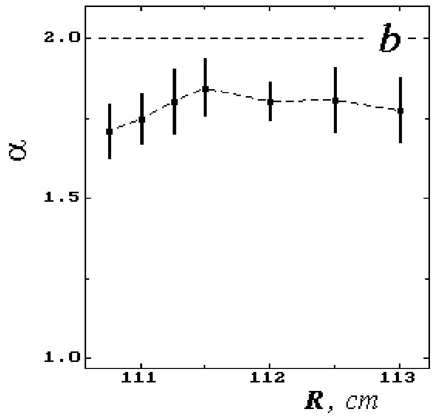
- **Time fractional, non-stationary media**

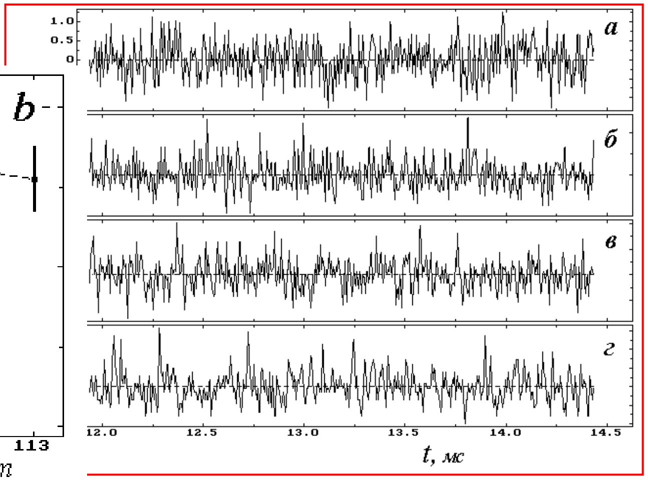
$$\frac{\partial^{\beta(t)}}{\partial t^{\beta(t)}} f(x,t) = K_{\beta}(t) \frac{\partial^2}{\partial x^2} f(x,t) \quad \Rightarrow \quad \left\langle x^2(t) \right\rangle \propto t^{\beta(t)} \quad ?$$

- **Space fractional: Lévy turbulence in plasma devices**









$$\frac{d\vec{r}}{dt} = \frac{\delta \vec{E}(\vec{r},t) \times \vec{B}_0}{B_0^2} \quad \Rightarrow \quad \frac{\partial f}{\partial t} = -D(\vec{r}) \left(-\Delta\right)^{\alpha(\vec{r})} f(\vec{r},t) \quad ?$$

Possibility 3. Diffusion equations with distributed order fractional derivative

C form

$$\int_a^b d\mu\, p(\mu) \frac{d^{\mu}}{dt^{\mu}} \phi(t)$$

R-L form

$$\int_a^b d\mu\, p(\mu) D_t^{\mu} \phi(t)$$

Riesz form

$$\int_a^b d\mu\, p(\mu) \frac{d^{\mu}}{d|x|^{\mu}} \phi(x)$$

$p(\mu) \geq 0$, $\int_a^b d\mu\, p(\mu) = 1$

Ordinary differential equations:

- Caputo form: generalizing stress-strain relation of inelastic media (M. Caputo, *Elasticita e Dissipazione*. Zanichelli Printer, Bologna,1969)
- R-L form with constant weight (Nakhushev, 1998)
- Ordinary diff equations containing sums of fractional derivatives (Podlubny, 1999)
- Distributed order eqs within functional calculus technique (Kochubei, 2008)
- Bagley and Torvik (2000), Diethelm and Ford (2001), numerical methods
- Hartley and Lorenzo, review (2002)

I. Natural Form of Distributed Order Time Fractional Diffusion Equation

Ch.,Gorenflo, Sokolov, 2002

$$\int_0^1 d\beta p(\beta) \frac{\partial^\beta f}{\partial t^\beta} = \frac{\partial^2 f}{\partial x^2}$$

(1) $p(\beta) \geq 0 \quad , \quad \int_0^1 d\beta p(\beta) = 1$

If $p(\beta) = \delta(\beta - \beta_0)$ ⇒ (mono)fractional diffusion equation

Solution of Eq(1) is a PDF

$$f(x,t) \quad \div \quad f(k,s) = \int_{-\infty}^{\infty} dx e^{ikx} \int_0^{\infty} dt e^{-st} f(x,t)$$

$$f(k,s) = \frac{1}{s} \frac{I_C(s)}{I_C(s) + k^2} \qquad (2)$$

$$I_C(s\tau) = \int_0^1 d\beta\, s^\beta p(\beta)$$

$$f(k,s) = \frac{I}{s} \int_0^\infty du\, e^{-u\left[I_C + k^2\right]} = \int_0^\infty du\, e^{-uk^2} \tilde{G}(u,s)$$, where $$\tilde{G}(u,s) = \frac{I_C(s)}{s} e^{-uI_C(s)}$$

$$f(x,t) = \int_0^\infty du \frac{e^{-x^2/(4\pi u)}}{\sqrt{4\pi u}} G(u,t) > 0$$

G(***u***,***t***) is a PDF providing subordination transformation from ***t*** to ***u***. Indeed,

1. $\tilde{G}(u,s)$ is completely monotonic

2 $\int_0^\infty du\, G(u,t) = 1$

Random process is subordinated to a Gaussian process using operational time

Fundamental solution in terms of Mellin-Barnes integral

Decelerating Subdiffusion: more anomalous in course of time

$$\int_0^1 d\beta\, p(\beta) \frac{\partial^\beta f}{\partial t^\beta} = \frac{\partial^2 f}{\partial x^2}$$

Ch, Gonchar, Gorenflo, Sokolov, 2003

Generic case $p(\beta) = B_1 \delta(\beta - \beta_1) + B_2 \delta(\beta - \beta_2) \quad , \quad \beta_1 < \beta_2$

$$\left\langle x^2(s) \right\rangle = \left(-\frac{\partial^2 f(k,s)}{\partial k^2} \right)_{k=0} \quad \Rightarrow \quad \left\langle x^2(t) \right\rangle \propto \begin{cases} t^{\beta_2} & , \quad t \to 0 \\ t^{\beta_1} & , \quad t \to \infty \end{cases}$$

2-parametric Mittag-Leffler

Tauberian theorems: small / large *s* → long / short *t*

Numerical simulation (Grünwald-Letnikov)

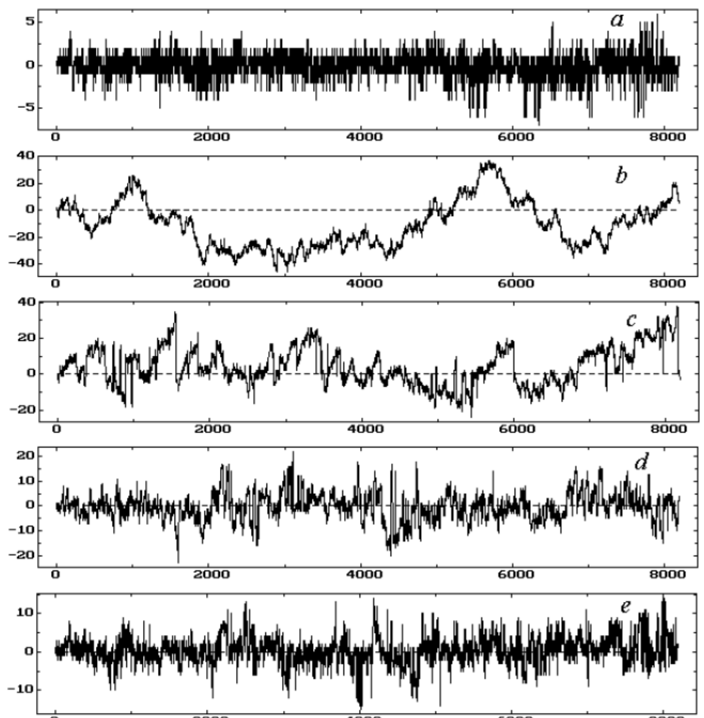
β_1 = 0.25 only

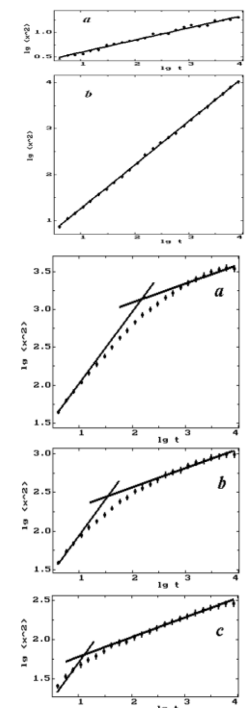
β_2 = 0.95 only

B1=0.03,
B2=0.97

B1=0.1,
B2=0.9

B1=0.3,
B2=0.7





Distributed order diffusion equation for superslow diffusion

HSC SM, Wroclaw

$$\int_0^1 d\beta\, \tau^{\beta-1} p(\beta) \frac{\partial^\beta f}{\partial t^\beta} = \frac{\partial^2 f}{\partial x^2}$$

with

$$p(\beta) = \nu \beta^{\nu-1}$$

$$\Rightarrow \left\langle x^2(t) \right\rangle \propto \ln^\nu t$$

Chechkin, Klafter, Sokolov, 2003

$$f(x,t) \simeq \exp\left[-\left(\frac{\Gamma(\nu+1)}{D\tau}\right)^{1/2} \frac{|x|}{\ln^{\nu/2}(t/\tau)}\right]$$

Laplace distribution

Relation to CTRW:

Extremely broad waiting time PDF :

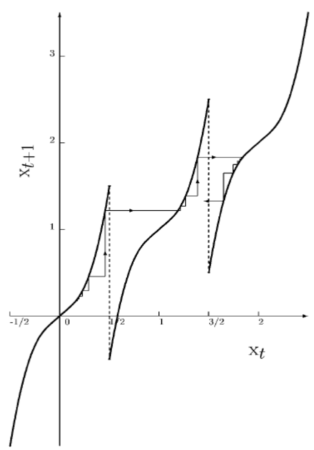
$$w(t) \propto \frac{1}{t\left[\log(t/\tau)\right]^{\nu+1}}$$

No moments

Havlin, Weiss (1990): disordered systems

Example: **iterated map (J. Drager, J. Klafter, 2000)**

$$x_{t+1} = x_t + a x_t^z \exp\left[-\left(\frac{b}{x_t}\right)^{z-1}\right], \quad z > 1$$



Aging, ergodicity breaking etc ???

Fractional Fokker-Planck equation for superslow diffusion

$$\int_0^1 d\beta \tau^{\beta-1} p(\beta) \partial^\beta f / \partial t^\beta = L_{FP} f(x,t) \quad f(x,0) = \delta(x)$$

$$L_{FP} = \frac{\partial}{\partial x} \frac{U'(x)}{m\gamma} + D \frac{\partial^2 f}{\partial x^2}$$

Separation ansatz: $f(x,t) = T(t)\varphi(x)$ ⇒ $T_n(t) \sim \dfrac{T_n(0)}{\lambda_n \tau \ln^\nu (t/\tau)}, \quad t \to \infty$ ⇒

Contrasts with Mittag-Leffler relaxation ~t^β

Difference from Sinai model

Einstein relation (contrasts with Sinai diffusion) ⇒

$$\langle x(t) \rangle_F = \frac{F \langle x^2(t) \rangle_0}{2 k_B T}$$

Interesting mathematical aspects of superslow diffusion equation: Meerschaert & Scheffler, 2005, 2006; Hanyga, 2007, Kochubei, 2008.

II. Modified Form of Distributed Order Time Fractional Diffusion Equation

$$\frac{\partial f}{\partial t} = \int_0^1 d\beta\, p(\beta) D_t^{1-\beta} \frac{\partial^2 f}{\partial x^2}$$

Thermodynamical interpretation

(no such for normal form)

$$\partial f / \partial t = -\partial j / \partial x$$

$$j(x,t) = \Phi_t \;\; \partial f(x,t) / \partial x$$

Flux dependent on the past history

Solution in F-L space

$$f(k,s) = \frac{I_{RL}}{s \;\; I_{RL} + k^2}$$

$$I_{RL}(s) = \left[\int_0^1 d\beta\, s^{-\beta} p(\beta) \right]^{-1}$$

No general proof of positivity

Accelerating subdiffusion

Generic case of two exponents :

$$p(\beta) = B_1 \delta(\beta - \beta_1) + B_2 \delta(\beta - \beta_2), \quad 0 < \beta_1 < \beta_2 \le 1$$

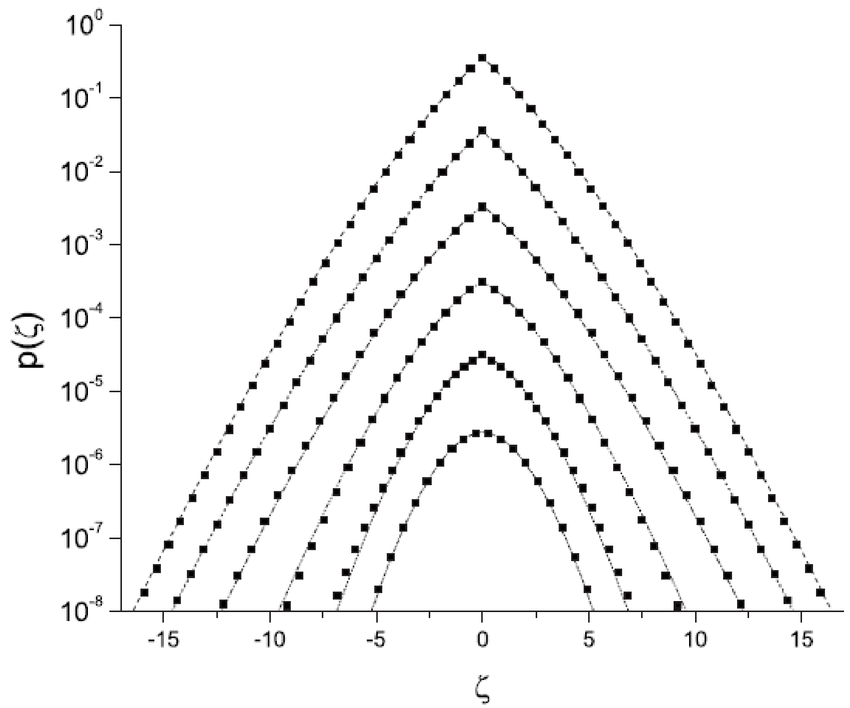
Positivity proved !

$$\langle x^2(t) \rangle \propto \begin{cases} t^{\beta_1} & , \quad short \quad times \\ t^{\beta_2} & , \quad long \quad times \end{cases}$$

Behavior is opposite to that for normal form demonstrating decelerating subdiffusion

PDF in terms of infinite series of the Fox functions

Numerical simulation of accelerating subdiffusion: approximation by single order solution in the whole time domain



Rescaled PDF for

t = 0.001; 0.01; 0.1; 1.0; 10; 100

• Solution of double order eq

with $\beta 1 = 0.5$; $\beta 2 = 1$

— Solution of single order eq

with βeff = 0.5; 0.5; 0.6; 0.73; 0.8; 0.95

Deff = 1; 1.05; 1.7; 1.95; 2.0; 1.4

The system governed by the distributed-order time-fractional diffusion equation has the properties very similar to the system whose exponent varies in time : $\beta = \beta(t) \Rightarrow$ similarity

to fractional diffusion equations of *variable order*

Distributed order time fractional diffusion equations and random walk models

Normal form

$$\int_0^1 d\beta\, p(\beta) \frac{\partial^\beta f}{\partial t^\beta} = K \frac{\partial^2 f}{\partial x^2}$$

Waiting time PDF:

(mono) fractional

$$w(\tau \mid \beta) \propto 1/\tau^{1+\beta}$$
$$0 < \beta < 1, \langle \tau \rangle = \infty$$

Normal form

$$w(\tau) = \int_0^1 d\beta\, p(\beta) w \;\; \tau \mid \beta$$

Mixture of waiting time PDFs: the longest waiting time survives at t $\rightarrow \infty$

$\Rightarrow$ Decelerating subdiffusion

Subordination form, Langevin description:

$$\frac{dx(s)}{ds} = \eta(s)\,, \; \frac{dt(s)}{ds} = \tau_1(s) + \tau_2(s) + \ldots$$

?

Distributed order time fractional diffusion equations and random walk models (continued)

$$\frac{\partial f}{\partial t} = \int_0^1 d\beta \, p(\beta) K D_t^{1-\beta} \frac{\partial^2 f}{\partial x^2}$$

Modified form

From Cox and Smith (1954):

“pooling” of renewal processes

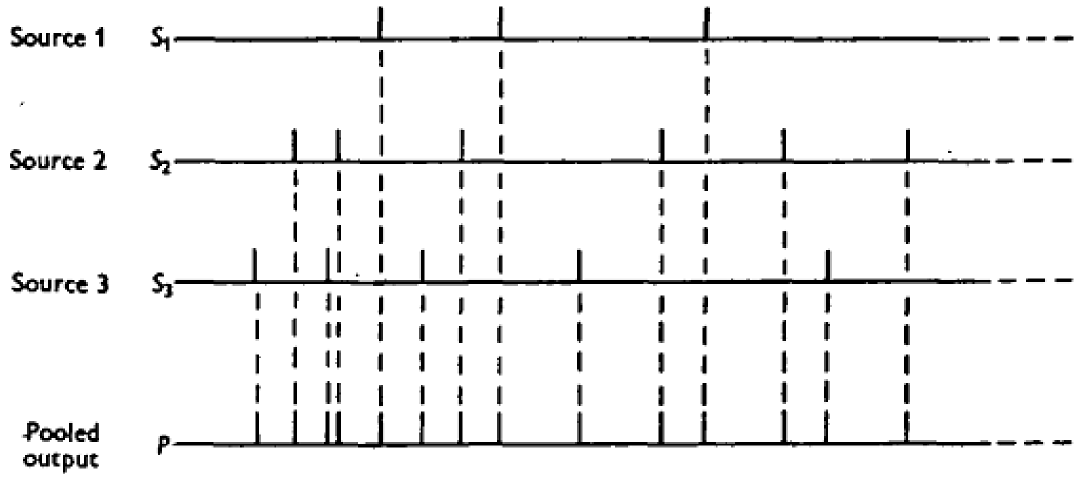


Fig. 1. The pooling of outputs.

Mixture “pooling” of CTRW processes:
The fastest survives at t → ∞
⇒ Accelerating subdiffusion

$$X(t) = \sum_{i=1}^{N} c_i X_i(t)$$

?

Subordination picture ?

Multifractal properties of underlying random processes

$$\left\langle |X(t)|^q \right\rangle = C(q) t^{\mu q}$$

Random fractal (self-affine) process, μ = const

In particular, the processes whose PDFs obey (mono)fractional diffusion equations:

$$\mu = \frac{\beta}{2}$$ Time fract

$$\mu = \frac{1}{\alpha}$$ Space fract

$$\left\langle |X(t)|^q \right\rangle = C(q) t^{\varphi(q)}$$

$\varphi(q)$ – nonlinear function
"standard" characterization of
non-self-affinity (multifractality)

$$\left\langle |X(t)|^q \right\rangle = C(q) t^{\varphi(q,t)}$$

Processes whose PDFs obey distributed order fractional diffusion equations

III. Natural Form of Distributed Order Space Fractional Diffusion Equation

$$\frac{\partial f}{\partial t} = \int_0^2 d\alpha\, p(\alpha) \frac{\partial^\alpha f}{\partial |x|^\alpha}, \quad p(x,0) = \delta(x)$$

Proved:

$f(x,t) > 0$

$$f(k,t) = \exp\ -t\int_0^2 d\alpha\, p(\alpha)\, |k|^\alpha$$

$$\int_{-\infty}^{\infty} dx\, f(x,t) = f(k=0,t) = 1$$

Double order case

$$p(\alpha) = A_1\delta(\alpha-\alpha_1) + A_2\delta(\alpha-\alpha_2)$$

$$0<\alpha_1<\alpha_2\le 2,\ A_1>0,\ A_2>0$$

$$f(k,t) = \exp\ -a_1|k|^{\alpha_1}t - a_2|k|^{\alpha_2}t$$

Lévy stable law

$$L_{\alpha,0}(k) = \exp\ -|k|^\alpha$$

$$f(x,t) = \frac{t^{-1/\alpha_1-1/\alpha_2}}{a_1^{1/\alpha_1}a_2^{1/\alpha_2}} \int_{-\infty}^{\infty} dx' L_{\alpha_1,0}\left(\frac{x-x'}{a_1 t^{1/\alpha_1}}\right) L_{\alpha_2,0}\left(\frac{x'}{a_2 t^{1/\alpha_2}}\right) > 0$$

Analogue of the second moment, $q < \alpha_1$, as a measure of diffusion properties

$$M_q(t;\alpha) = \left\langle |x|^q \right\rangle^{2/q} \propto \begin{cases} t^{2/\alpha_2}, & t\to 0 \\ t^{2/\alpha_1}, & t\to\infty \end{cases}$$

$$1/\alpha_1 > 1/\alpha_2$$

Natural form leads to *accelerating* superdiffusion

Interesting application: together with A. Iomin (in progress)

IV. Modified Form of Distributed Order Space Fractional Diffusion Equation

$$\int_0^2 d\alpha\, p(\alpha) \frac{\partial^{2-\alpha}}{\partial |x|^{2-\alpha}} \frac{\partial f}{\partial t} = -\frac{\partial^2 f}{\partial x^2}, \quad f(x,0) = \delta(x)$$

,

Characteristic function :

$$f(k,t) = \exp\left\{ -\frac{t}{\int_0^2 d\alpha\, p(\alpha) |k|^\alpha} \right\}$$

$$\int_{-\infty}^{\infty} dx\, f(x,t) = f(k=0,t) = 1$$

No general proof of positivity

Double order case

$$p(\alpha) = A_1 \delta(\alpha - \alpha_1) + A_2 \delta(\alpha - \alpha_2)$$

$$0 < \alpha_1 < \alpha_2 \le 2, \ A_1 > 0, \ A_2 > 0$$

proved : $f(x,t) > 0$

$$M_q(t;\alpha) = \left\langle |x|^q \right\rangle^{2/q} \propto \begin{cases} t^{2/\alpha_1}, & t \to 0 \\ t^{2/\alpha_2}, & t \to \infty \end{cases}$$

⇒

Modified form *leads to decelerating* superdiffusion, in contrast to natural form

Now: Take particular case $\alpha_2 = 2$

Sokolov, Ch, Klafter, Acta Phys Polonica 2004

Fractional Diffusion Equation for Power Law Truncated Lévy Process

(observation: Stanley's group (2003): PDF of commodity prices;
Cohen, Venkatesh (2006): database of S&P index)

Sokolov, Ch, Klafter, 2005

- **"PLT LFs" : PDF resembles Lévy stable distribution in the central part**
- **at greater scales the asymptotics decay in a power-law way, but faster, than the Lévy stable ones, therefore,** $\langle x^2 \rangle < \infty$ $\Rightarrow$ **the Central Limit Theorem is applied** $\Rightarrow$
- **at large times the PDF tends to Gaussian, however, *very slowly***

Governing equation: Particular case of the modified form of *distributed order space FDE*:

$$\left(1 - C_\alpha \frac{\partial^{2-\alpha}}{\partial |x|^{2-\alpha}}\right) \frac{\partial f(x,t)}{\partial t} = D \frac{\partial^2 f(x,t)}{\partial x^2}$$

$$0 < \alpha < 2$$

$$f(x,t) \simeq \frac{\Gamma(5-\alpha) \sin \pi\alpha/2}{\pi} \frac{D C_\alpha t}{x^{5-\alpha}}, \quad x \to \infty$$

At small times the Lévy distribution is truncated by a power law with a power between 3 and 5. Due to the finiteness of the second moment the PDF *f(x,t)* slowly converges to a Gaussian

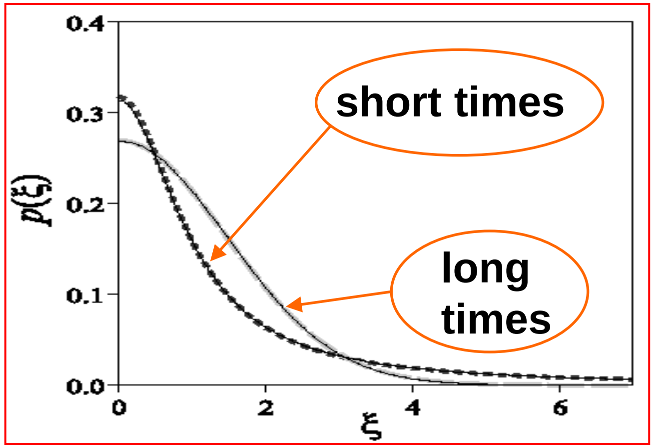
Power-law truncated Lévy flights: Probability to stay at the origin

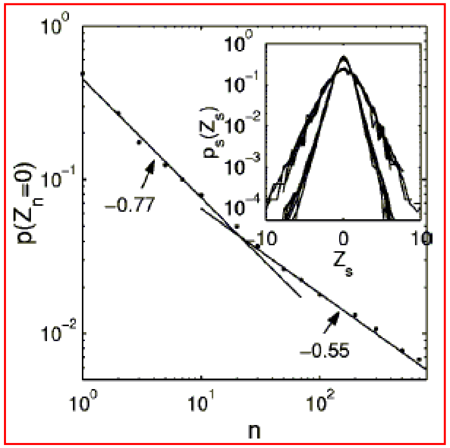
$$LFs: \; f(0,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-|k|^{\alpha} t} \propto t^{-1/\alpha} \quad , \quad 0 < \alpha \le 2$$

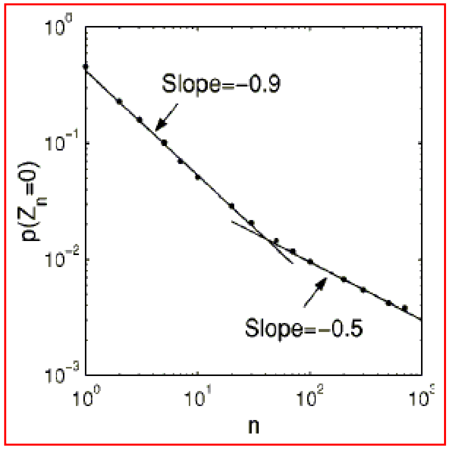
The slope $-1/\alpha$ in the double logarithmic scale changes into -1/2 for the Gaussian

Theory, $\alpha = 1$

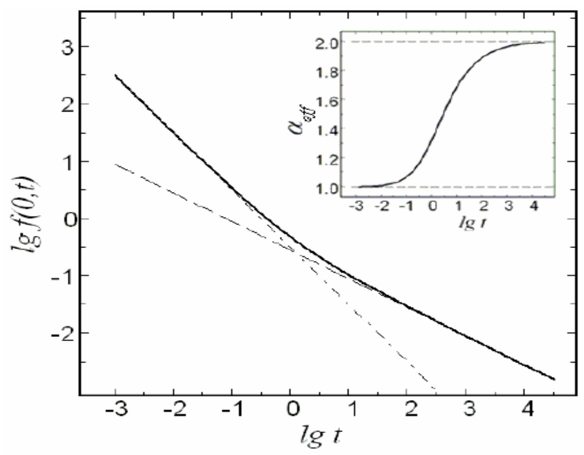
Experiment on the ADITYA tokamak

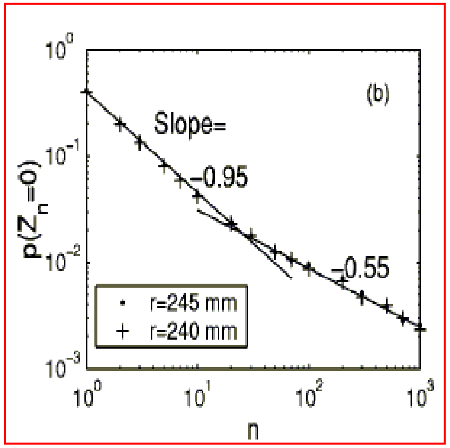


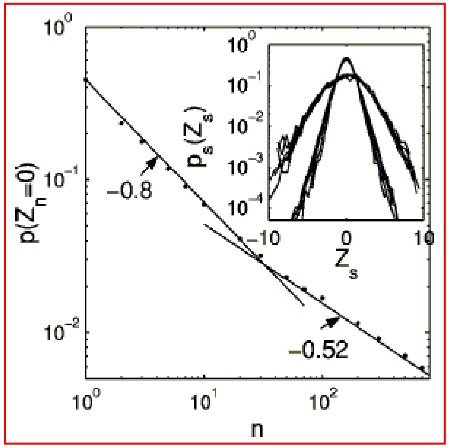




Probability to stay at the origin







Experiment on the ADITYA: probabilities of return to the origin for different radial positions of the probes [R.Jha et al. Phys. Plasmas.2003.Vol.10.No.3.PP.699-704]. Insets: rescaled PDFs.

Power-law truncated Lévy flights: Fractional-order moments

- **“Normal” behavior of the 2nd moment for PLT LFs**

$$\left\langle x^2(t) \right\rangle = - \frac{\partial^2 f(k,t)}{\partial k^2}\bigg|_{k=0} = 2D\,t \qquad (1)$$

Ch, Gonchar, Gorenflo, Korabel, Sokolov, 2008

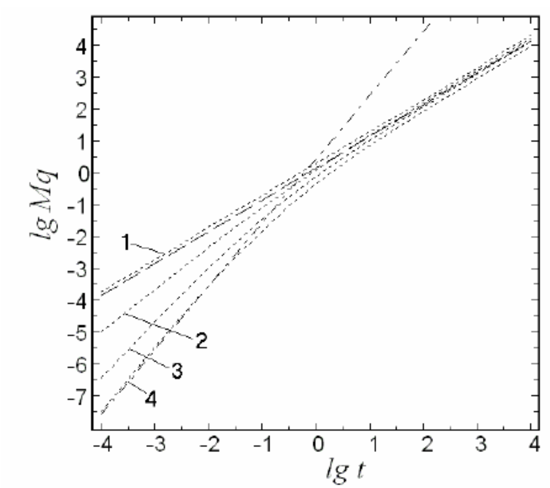
- **Analogue of the 2nd moment for LFs to characterize super-diffusive behavior, $q < \alpha$**

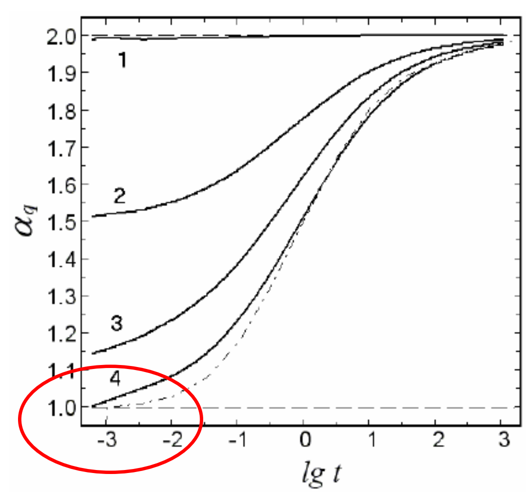
$$M_q\ t;\alpha \ \equiv \left\langle |x\ t|^q \right\rangle^{2/q} \propto t^{2/\alpha} : \quad \textit{faster than} \quad t^1 \quad , q < \alpha < 2 \qquad (2)$$

Superdiffusive behavior of fractional moments. Example: $\alpha = 1$.

1. Fractional moments for LFs:

1 : q = 2.0
2: q = 1.5
3 : q = 1.0
4 : q = 0.5





“effective” index :

$$\alpha_q(t) = 2\left(\frac{d\ \lg M_q\ t;\alpha}{d(\lg t)} \right)^{-1}$$

Left: *Mq* versus time in log-log scale for *q* = 2.0, 1.5, 1.0, and 0.5 (lines 1, 2, 3 and 4, respectively). Right: the quantity $\alpha_q(t)$ for the same *q* values as in the left panel.

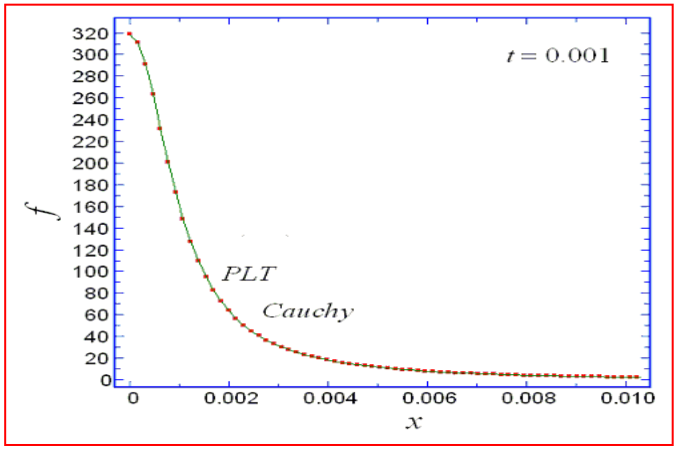
Power-law truncated LFs: Evolution of the PDF from truncated Cauchy (α = 1) to the Gaussian with a power-law tail

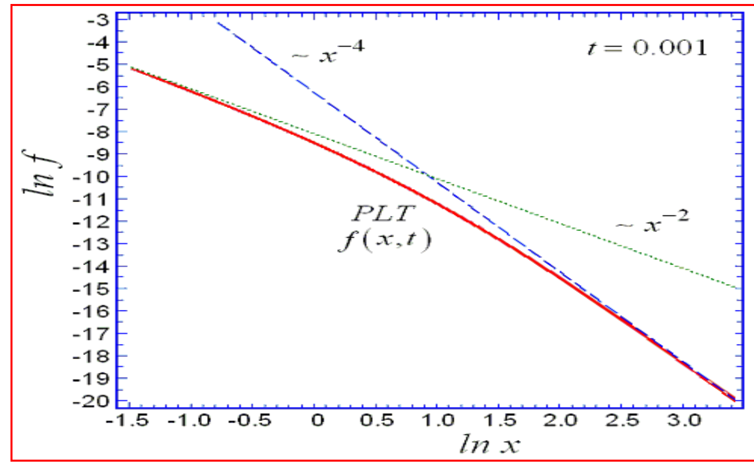
Ch, Gonchar, Gorenflo, Korabel, Sokolov, 2008

linear scale

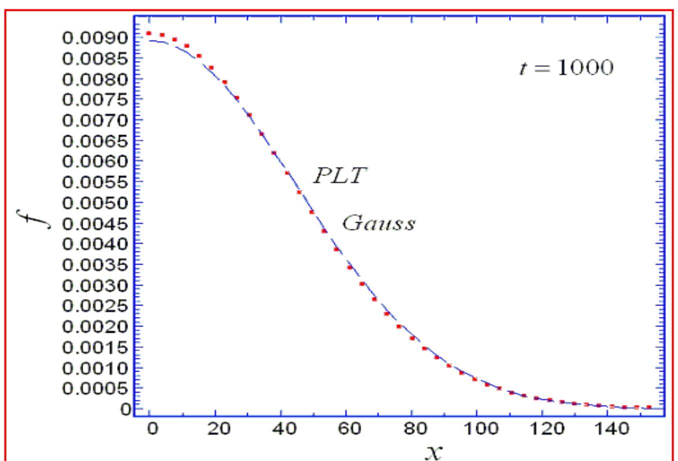
log-log scale

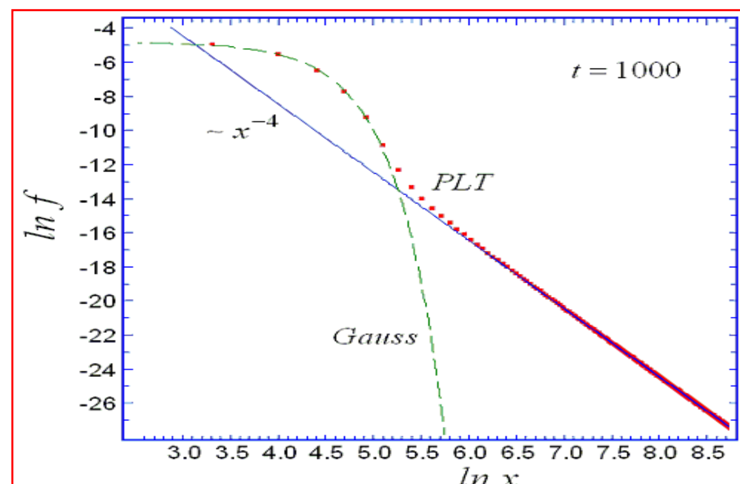
PDF
at short time
(= 0.001)





PDF
at long time
(= 1000)





Conclusions

	normal form	modified form
subdiffusion	$\int_0^1 d\beta\, p(\beta) \frac{\partial^\beta f}{\partial t^\beta} = K \frac{\partial^2 f}{\partial x^2}$ Decelerating subdiffusion and superslow diffusion	$\frac{\partial f}{\partial t} = \int_0^1 d\beta\, p(\beta) K D_t^{1-\beta} \frac{\partial^2 f}{\partial x^2}$ Accelerating subdiffusion
superdiffusion	$\frac{\partial f}{\partial t} = \int_0^2 d\alpha\, p(\alpha) K \frac{\partial^\alpha f}{\partial \lvert x\rvert^\alpha}$ Accelerating superdiffusion	$\int_0^2 d\alpha\, p(\alpha) \frac{\partial^{2-\alpha}}{\partial \lvert x\rvert^{2-\alpha}} \frac{\partial f}{\partial t} = -K \frac{\partial^2 f}{\partial x^2}$ Decelerating super-diffusion and power law truncated Lévy flights

Might be useful for description of the different anomalous diffusion phenomena demonstrating non-scaling behavior