INFINITY

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Foreword by Dr. Geoff Smith

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Foreword

The International Mathematical Olympiad is the largest and most prestigious mathematics competition in the world. It is held each July, and the host city changes from year to year. It has existed since 1959.

Originally it was a competition between students from a small group of communist countries, but by the late 1960s, social-democratic nations were starting to send teams. Over the years the enthusiasm for this competition has built up so much that very soon (I write in 2008) there will be an IMO with students participating from over 100 countries. In recent years, the format has become stable. Each nation can send a team of up to six students. The students compete as individuals, and must try to solve 6 problems in 9 hours of examination time, spread over two days.

The nations which do consistently well at this competition must have at least one (and probably at least two) of the following attributes:

(a) A large population.
(b) A significant proportion of its population in receipt of a good education.
(c) A well-organized training infrastructure to support mathematics competitions.
(d) A culture which values intellectual achievement.

Alternatively, you need a cloning facility and a relaxed regulatory framework.

Mathematics competitions began in the Austro-Hungarian Empire in the 19^{th} century, and the IMO has stimulated people into organizing many other related regional and world competitions. Thus there are quite a few opportunities to take part in international mathematics competitions other than the IMO.

The issue arises as to where talented students can get help while they prepare themselves for these competitions. In some countries the students are lucky, and there is a well-developed training regime. Leaving aside the coaching, one of the most important features of these regimes is that they put talented young mathematicians together. This is very important, not just because of the resulting exchanges of ideas, but also for mutual encouragment in a world where interest in mathematics is not always widely understood. There are some very good books available, and a wealth of resources on the internet, including this excellent book Infinity.

The principal author of Infinity is Hojoo Lee of Korea. He is the creator of many beautiful problems, and IMO juries have found his style most alluring. Since 2001 they have chosen 8 of his problems for IMO papers. He has some way to go to catch up with the sage of Scotland, David Monk, who has had 14 problems on IMO papers. These two gentlemen are reciprocal Nemeses, dragging themselves out of bed every morning to face the possibility that the other has just had a good idea. What they each need is a framed picture of the other, hung in their respective studies. I will organize this.

The other authors of Infinity are the young mathematicians Tom Lovering of the United Kingdom and Cosmin Pohoaţă of Romania. Tom is an alumnus of the UK IMO team, and is now starting to read mathematics at Newton's outfit, Trinity College Cambridge. Cosmin has a formidable internet presence, and is a PEN activist (Problems in Elementary Number theory).

One might wonder why anyone would spend their time doing mathematics, when there are so many other options, many of which are superficially more attractive. There are a whole range of opportunities for an enthusiastic Sybarite, ranging from full scale debauchery down to gentle dissipation. While not wishing to belittle these interesting hobbies, mathematics can be more intoxicating.

There is danger here. Many brilliant young minds are accelerated through education, sometimes graduating from university while still under 20. I can think of people for whom this has worked out well, but usually it does not. It is not sensible to deprive teenagers of the company of their own kind. Being a teenager is very stressful; you have to cope with hormonal poisoning, meagre income, social incompetance and the tyranny of adults. If you find yourself with an excellent mathematical mind, it just gets worse, because you have to endure the approval of teachers.

Olympiad mathematics is the sensible alternative to accelerated education. Why do lots of easy courses designed for older people, when instead you can do mathematics which is off the contemporary mathematics syllabus because it is too interesting and too hard? Euclidean and projective geometry and the theory of inequalities (laced with some number theory and combinatorics) will keep a bright young mathematician intellectually engaged, off the streets, and able to go school discos with other people in the same unfortunate teenaged state.

The authors of Infinity are very enthusiastic about MathLinks, a remarkable internet site. While this is a fantastic resource, in my opinion the atmosphere of the Olympiad areas is such that newcomers might feel a little overwhelmed by the extraordinary knowledge and abilities of many of the people posting. There is a kinder, gentler alternative in the form of the nRich site based at the University of Cambridge. In particular the Onwards and Upwards section of their Ask a Mathematician service is MathLinks for herbivores. While still on the theme of material for students at the beginning of their maths competition careers, my accountant would not forgive me if I did not mention A Mathematical Olympiad Primer available on the internet from the United Kingdom Mathematics Trust, and also through the Australian Mathematics Trust.

Returning to this excellent weblished document, Infinity is an wonderful training resource, and is brim full of charming problems and exercises. The mathematics competition community owes the authors a great debt of gratitude.

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Overture

It was a dark decade until MathLinks was born. However, after Valentin Vornicu founded MathLinks, everything has changed. As the best on-line community, MathLinks helps young students around the worlds to develop problem-solving strategies and broaden their mathematical backgrounds. Nowadays, students, as young mathematicians, use the LaTeX typesetting system to upload recent olympiad problems or their own problems and enjoy mathematical friendship by sharing their creative solutions with each other. In other words, MathLinks encourages and challenges young people in all countries, foster friendships between young mathematicians around the world. Yes, it exactly coincides with the aim of the IMO. Actually, MathLinks is even better than IMO. Simply, it is because everyone can join MathLinks!

In this never-ending project, which bears the name Infinity, we offer a delightful playground for young mathematicians and try to continue the beautiful spirit of IMO and MathLinks. Infinity begins with a chapter on elementary number theory and mainly covers Euclidean geometry and inequalities. We re-visit beautiful well-known theorems and present heuristics for elegant problem-solving. Our aim in this weblication is not just to deliver *must-know* techniques in problem-solving. Young readers should keep in mind that our aim in this project is to present the beautiful aspects of Mathematics. Eventually, Infinity will admit bridges between Olympiads Mathematics and undergraduate Mathematics.

Here goes the reason why we focus on the algebraic and trigonometric methods in geometry. It is a *cliché* that, in the IMOs, some students from hard-training countries used to employ the brute-force algebraic techniques, such as employing trigonometric methods, to attack hard problems from classical triangle geometry or to trivialize easy problems. Though MathLinks already has been contributed to the distribution of the power of algebraic methods, it seems that still many people do not feel the importance of such techniques. Here, we try to destroy such situations and to deliver a friendly introduction on algebraic and trigonometric methods in geometry.

We have to confess that many materials in the first chapter are stolen from PEN (Problems in Elementary Number theory). Also, the lecture note on inequalities is a continuation of the weblication TIN (Topic in INequalities). We are indebted to Orlando Döhring and Darij Grinberg for providing us with TeX files including collections of interesting problems. We owe great debts to Stanley Rabinowitz who kindly sent us his paper. We'd also like to thank Marian Muresan for his excellent collection of problems. We are pleased that Cao Minh Quang sent us various Vietnam problems and nice proofs of Nesbitt's Inequality.

Infinity is a joint work of three coauthors: Hojoo Lee (Korea), Tom Lovering (United Kingdom), and Cosmin Pohoaţă (Romania). We would greatly appreciate hearing about comments and corrections from our readers. Have fun!

Contents

1. Number Theory

> Why are numbers beautiful? It's like asking why is Beethoven's Ninth Symphony beautiful. If you don't see why, someone can't tell you. I know numbers are beautiful. If they aren't beautiful, nothing is.
>
> \- P. Erdős

1.1. **Fundamental Theorem of Arithmetic.**

In this chapter, we meet various inequalities and estimations which appears in number theory. Throughout this section, we denote $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ the set of positive integers, integers, rational numbers, respectively. For integers a and b, we write $a \mid b$ if there exists an integer k such that $b = ka$. Our starting point In this section is the cornerstone theorem that every positive integer $n \neq 1$ admits a unique factorization of prime numbers.

Theorem 1.1. (The Fundamental Theorem of Arithmetic in $\mathbb{N}$) Let $n \neq 1$ be a positive integer. Then, n is a product of primes. If we ignore the order of prime factors, the factorization is unique. Collecting primes from the factorization, we obtain a standard factorization of n:

$$n = {p_1}^{e_1} \cdots {p_l}^{e_l}.$$

The distinct prime numbers $p_1, \cdots, p_l$ and the integers $e_1, \cdots, e_l \geq 0$ are uniquely determined by n.

We define $ord_p(n)$, the order of $n \in \mathbb{N}$ at a prime p,[1] by the nonnegative integer k such that $p^k \mid n$ but $p^{k+1} \nmid n$. Then, the standard factorization of positive integer n can be rewritten as the form

$$n = \prod_{\text{p : prime}} p^{ord_p(n)}.$$

One immediately has the following simple and useful criterion on divisibility.

Proposition 1.1. *Let A and B be positive integers. Then, A is a multiple of B if and only if the inequality*

$$ord_p(A) \geq ord_p(B)$$

holds for all primes p.

Epsilon 1. [NS] Let a and b be positive integers such that

$$a^k \mid b^{k+1}$$

for all positive integers k. Show that b is divisible by a.

We now employ a formula for the prime factorization of $n!$. Let $\lfloor x \rfloor$ denote the largest integer smaller than or equal to the real number x.

Delta 1. (De Polignac's Formula) Let p be a prime and let n be a nonnegative integer. Then, the largest exponent e of $n!$ such that $p^e \mid n!$ is given by

$$ord_p(n!) = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor.$$

[1] Here, we do not assume that $n \neq 1$.

Example 1. *Let $a_1, \cdots, a_n$ be nonnegative integers. Then, $(a_1 + \cdots + a_n)!$ is divisible by $a_1! \cdots a_n!$.*

Proof. Let p be a prime. Our job is to establish the inequality

$$ord_p\left(\, (a_1 + \cdots + a_n)!\, \right) \geq ord_p\left(a_1!\right) + \cdots ord_p\left(a_n!\right).$$

or

$$\sum_{k=1}^{\infty} \left\lfloor \frac{a_1 + \cdots + a_n}{p^k} \right\rfloor \geq \sum_{k=1}^{\infty} \left(\left\lfloor \frac{a_1}{p^k} \right\rfloor + \cdots + \left\lfloor \frac{a_n}{p^k} \right\rfloor \right).$$

However, the inequality

$$\lfloor x_1 + \cdots + x_n \rfloor \geq \lfloor x_1 \rfloor + \cdots \lfloor x_n \rfloor,$$

holds for all real numbers $x_1, \cdots, x_n$. □

Epsilon 2. [IMO 1972/3 UNK] *Let m and n be arbitrary non-negative integers. Prove that*

$$\frac{(2m)!(2n)!}{m!n!(m+n)!}$$

is an integer.

Epsilon 3. *Let $n \in \mathbb{N}$. Show that $\mathcal{L}_n := lcm(1, 2, \cdots, 2n)$ is divisible by $\mathcal{K}_n := \binom{2n}{n} = \frac{(2n)!}{(n!)^2}$.*

Delta 2. (Canada 1987) Show that, for all positive integer n,

$$\lfloor \sqrt{n} + \sqrt{n+1} \rfloor = \lfloor \sqrt{4n+1} \rfloor = \lfloor \sqrt{4n+2} \rfloor = \lfloor \sqrt{4n+3} \rfloor.$$

Delta 3. (Iran 1996) Prove that, for all positive integer n,

$$\lfloor \sqrt{n} + \sqrt{n+1} + \sqrt{n+2} \rfloor = \lfloor \sqrt{9n+8} \rfloor.$$

1.2. **Fermat's Infinite Descent.** In this section, we learn Fermat's trick, which bears the name method of infinite descent. It is extremely useful for attacking many Diophantine equations. We first present a proof of Fermat's Last Theorem for $n = 4$.

Theorem 1.2. (The Fermat-Wiles Theorem) Let $n \geq 3$ be a positive integer. The equation

$$x^n + y^n = z^n$$

has no solution in positive integers.

Lemma 1.1. *Let σ be a positive integer. If we have a factorization $\sigma^2 = AB$ for some relatively integers A and B, then the both factors A and B are also squares. There exist positive integers a and b such that*

$$\sigma = ab, \quad A = a^2, \quad B = b^2, \quad gcd(a, b) = 1.$$

Proof. Use The Fundamental Theorem of Arithmetic. □

Lemma 1.2. (Primitive Pythagoras Triangles) Let $x, y, z \in \mathbb{N}$ with $x^2 + y^2 = z^2$, $gcd(x, y) = 1$, and $x \equiv 0 \ (mod\ 2)$ Then, there exists positive integers p and q such that $gcd(p, q) = 1$ and

$$(x,\ y,\ z) = \left(2pq,\ p^2 - q^2,\ p^2 + q^2\right).$$

Proof. The key observation is that the equation can be rewritten as

$$\left(\frac{x}{2}\right)^2 = \left(\frac{z+y}{2}\right)\left(\frac{z-y}{2}\right).$$

Reading the equation $x^2 + y^2 = z^2$ modulo 2, we see that both y and z are odd. Hence, $\frac{z+y}{2}$, $\frac{z-y}{2}$, and $\frac{x}{2}$ are positive integers. We also find that $\frac{z+y}{2}$ and $\frac{z-y}{2}$ are relatively prime. Indeed, if $\frac{z+y}{2}$ and $\frac{z-y}{2}$ admits a common prime divisor p, then p also divides both $y = \frac{z+y}{2} - \frac{z-y}{2}$ and $\left(\frac{x}{2}\right)^2 = \left(\frac{z+y}{2}\right)\left(\frac{z-y}{2}\right)$, which means that the prime p divides both x and y. This is a contradiction for $gcd(x, y) = 1$. Now, applying the above lemma, we obtain

$$\left(\frac{x}{2},\ \frac{z+y}{2},\ \frac{z-y}{2}\right) = \left(pq,\ p^2,\ q^2\right)$$

for some positive integers p and q such that $\gcd(p, q) = 1$. □

Theorem 1.3. *The equation $x^4 + y^4 = z^2$ has no solution in positive integers.*

Proof. Assume to the contrary that there exists a bad triple (x, y, z) of positive integers such that $x^4 + y^4 = z^2$. Pick a bad triple $(A, B, C) \in \mathcal{D}$ so that $A^4 + B^4 = C^2$. Letting d denote the greatest common divisor of A and B, we see that $C^2 = A^4 + B^4$ is divisible by d^4, so that C is divisible by d^2. In the view of $\left(\frac{A}{d}\right)^4 + \left(\frac{B}{d}\right)^4 = \left(\frac{C}{d^2}\right)^2$, we find that $(a, b, c) = \left(\frac{A}{d}, \frac{B}{d}, \frac{C}{d^2}\right)$ is also in $\mathcal{D}$, that is,

$$a^4 + b^4 = c^2.$$

Furthermore, since d is the greatest common divisor of A and B, we have $\gcd(a, b) = \gcd\left(\frac{A}{d}, \frac{B}{d}\right) = 1$. Now, we do the parity argument. If both a and b are odd, we find that $c^2 \equiv a^4 + b^4 \equiv 1 + 1 \equiv 2 \ (mod\ 4)$, which is impossible. By symmetry, we may assume that a is even and that b is odd. Combining results, we see that a^2 and b^2

are relatively prime and that a^2 is even. Now, in the view of $\left(a^2\right)^2 + \left(b^2\right)^2 = c^2$, we obtain

$$\left(a^2,\ b^2,\ c\right) = \left(2pq,\ p^2 - q^2,\ p^2 + q^2\right).$$

for some positive integers p and q such that $\gcd(p, q) = 1$. It is clear that p and q are of opposite parity. We observe that

$$q^2 + b^2 = p^2.$$

Since b is odd, reading it modulo 4 yields that q is even and that p is odd. If q and b admit a common prime divisor, then $p^2 = q^2 + b^2$ guarantees that p also has the prime, which contradicts for $\gcd(p, q) = 1$. Combining the results, we see that q and b are relatively prime and that q is even. In the view of $q^2 + b^2 = p^2$, we obtain

$$(q,\ b,\ p) = \left(2mn,\ m^2 - n^2,\ m^2 + n^2\right).$$

for some positive integers m and n such that $\gcd(m, n) = 1$. Now, recall that $a^2 = 2pq$. Since p and q are relatively prime and since q is even, it guarantees the existence of the pair (P, Q) of positive integers such that

$$a = 2PQ,\ p = P^2,\ q = 2Q^2,\ \gcd(P, Q) = 1.$$

It follows that $2Q^2 = 2q = 2mn$ so that $Q = mn$. Since $\gcd(m, n) = 1$, this guarantees the existence of the pair (M, N) of positive integers such that

$$Q = MN,\ m = M^2,\ n = N^2,\ \gcd(M, N) = 1.$$

Combining the results, we find that $P^2 = p = m^2 + n^2 = M^4 + N^4$ so that (M, N, P) is a bad triple. Recall the starting equation $A^4 + B^4 = C^2$. Now, let's summarize up the results what we did. The bad triple (A, B, C) produces a new bad triple (M, N, P). However, we need to check that it is indeed new. We observe that $P < C$. Indeed, we deduce

$$P \le P^2 = p < p^2 + q^2 = c = \frac{C}{d^2} \le C.$$

In words, from a solution of $x^4 + y^4 = z^2$, we are able to find another solution with smaller positive integer z. The key point is that this reducing process can be repeated. Hence, it produces to an infinite sequence of strictly decreasing positive integers. However, it is clearly impossible. We therefore conclude that there exists no bad triple. □

Corollary 1.1. *The equation $x^4 + y^4 = z^4$ has no solution in positive integers.*

Proof. Letting $w = z^2$, we obtain $x^4 + y^4 = w^2$. □

We now include a recent problem from IMO as another working example.

Example 2. [IMO 2007/5 IRN] *Let a and b be positive integers. Show that if $4ab - 1$ divides $\left(4a^2 - 1\right)^2$, then $a = b$.*

First Solution. (by NZL at IMO 2007) When $4ab - 1$ divides $\left(4a^2 - 1\right)^2$ for two distinct positive integers a and b, we say that (a, b) is a bad pair. We want to show that there is no bad pair. Suppose that $4ab - 1$ divides $\left(4a^2 - 1\right)^2$. Then, $4ab - 1$ also divides

$$b\left(4a^2 - 1\right)^2 - a\left(4ab - 1\right)\left(4a^2 - 1\right) = (a - b)\left(4a^2 - 1\right).$$

The converse also holds as $\gcd(b, 4ab-1) = 1$. Similarly, $4ab-1$ divides $(a-b)\left(4a^2-1\right)^2$ if and only if $4ab-1$ divides $(a-b)^2$. So, the original condition is equivalent to the condition

$$4ab - 1 \mid\ (a-b)^2 .$$

This condition is symmetric in a and b, so (a, b) is a bad pair if and only if (b, a) is a bad pair. Thus, we may assume without loss of generality that $a > b$ and that our bad pair of this type has been chosen with the smallest possible vales of its first element. Write $(a-b)^2 = m(4ab-1)$, where m is a positive integer, and treat this as a quadratic in a:

$$a^2 + (-2b - 4ma)a + \left(b^2 + m\right) = 0.$$

Since this quadratic has an integer root, its discriminant

$$(2b + 4mb)^2 - 4\left(b^2 + m\right) = 4\left(4mb^2 + 4m^2b^2 - m\right)$$

must be a perfect square, so $4mb^2 + 4m^2b^2 - m$ is a perfect square. Let his be the square of $2mb + t$ and note that $0 < t < b$. Let $s = b - t$. Rearranging again gives:

$$\begin{aligned} 4mb^2 + 4m^2b^2 - m &= (2mb + t)^2 \\ m\left(4b^2 - 4bt - 1\right) &= t^2 \\ m\left(4b^2 - 4b(b-s) - 1\right) &= (b-s)^2 \\ m(4bs - 1) &= (b-s)^2. \end{aligned}$$

Therefore, (b, s) is a bad pair with a smaller first element, and we have a contradiction. $\square$

Second Solution. (by UNK at IMO 2007) This solution is inspired by the solution of NZL7 and Atanasov's special prize solution at IMO 1988 in Canberra. We begin by copying the argument of NZL7. A counter-example (a, b) is called a bad pair. Consider a bad pair (a, b) so $4ab - 1 | \left(4a^2-1\right)^2$. Notice that $b\left(4a^2-1\right) - (4ab-1)a = a - b$ so working modulo $4ab-1$ we have $b^2\left(4a^2-1\right) \equiv (a-b)^2$. Now, b^2 an $4ab-1$ are coprime so $4ab-1$ divides $\left(4a^2-1\right)^2$ if and only if $4ab-1$ divides $(a-b)^2$. This condition is symmetric in a and b, so we learn that (a, b) is a bad pair if and only if (b, a) is a bad pair. Thus, we may assume that $a > b$ and we may as well choose a to be minimal among all bad pairs where the first component is larger than the second. Next, we deviate from NZL7's solution. Write $(a-b)^2 = m(4ab-1)$ and treat it as a quadratic so a is a root of

$$x^2 + (-2b - 4mb)x + \left(b^2 + m\right) = 0.$$

The other root must be an integer c since $a + c = 2b + 4mb$ is an integer. Also, $ac = b^2 + m > 0$ so c is positive. We will show that $c < b$, and then the pair (b, c) will violate the minimality of (a, b). It suffices to show that $2b + 4mb < a + b$, i.e., $4mb < a - b$. Now,

$$4b(a-b)^2 = 4mb(4ab-1)$$

so it suffices to show that $4b(a-b) < 4ab - 1$ or rather $1 < 4b^2$ which is true. $\square$

Delta 4. [IMO 1988/6 FRG] *Let a and b be positive integers such that $ab+1$ divides a^2+b^2. Show that*

$$\frac{a^2 + b^2}{ab + 1}$$

is the square of an integer.

Delta 5. (Canada 1998) Let m be a positive integer. Define the sequence $\{a_n\}_{n\geq 0}$ by

$$a_0 = 0,\ a_1 = m,\ a_{n+1} = m^2 a_n - a_{n-1}.$$

Prove that an ordered pair (a, b) of non-negative integers, with $a \leq b$, gives a solution to the equation

$$\frac{a^2 + b^2}{ab + 1} = m^2$$

if and only if (a, b) is of the form (a_n, a_{n+1}) for some $n \geq 0$.

Delta 6. *Let x and y be positive integers such that xy divides $x^2 + y^2 + 1$. Show that*

$$\frac{x^2 + y^2 + 1}{xy} = 3.$$

Delta 7. *Find all triple (x, y, z) of integers such that*

$$x^2 + y^2 + z^2 = 2xyz.$$

Delta 8. (APMO 1989) Prove that the equation

$$6\left(6a^2 + 3b^2 + c^2\right) = 5n^2$$

has no solutions in integers except $a = b = c = n = 0$.

1.3. **Monotone Multiplicative Functions.** In this section, we study when multiplicative functions has the monotonicity.

Example 3. (Canada 1969) Let $\mathbb{N} = \{1, 2, 3, \cdots\}$ denote the set of positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for all $m, n \in \mathbb{N}$: $f(2) = 2$, $f(mn) = f(m)f(n)$, $f(n+1) > f(n)$.

First Solution. We first evaluate $f(n)$ for small n. It follows from $f(1 \cdot 1) = f(1) \cdot f(1)$ that $f(1) = 1$. By the multiplicity, we get $f(4) = f(2)^2 = 4$. It follows from the inequality $2 = f(2) < f(3) < f(4) = 4$ that $f(3) = 3$. Also, we compute $f(6) = f(2)f(3) = 6$. Since $4 = f(4) < f(5) < f(6) = 6$, we get $f(5) = 5$. We prove by induction that $f(n) = n$ for all $n \in \mathbb{N}$. It holds for $n = 1, 2, 3$. Now, let $n > 2$ and suppose that $f(k) = k$ for all $k \in \{1, \cdots, n\}$. We show that $f(n+1) = n+1$.

Case 1. $n+1$ is composite. One may write $n+1 = ab$ for some positive integers a and b with $2 \le a \le b \le n$. By the inductive hypothesis, we have $f(a) = a$ and $f(b) = b$. It follows that $f(n+1) = f(a)f(b) = ab = n+1$.

Case 2. $n+1$ is prime. In this case, $n+2$ is even. Write $n+2 = 2k$ for some positive integer k. Since $n \ge 2$, we get $2k = n+2 \ge 4$ or $k \ge 2$. Since $k = \frac{n+2}{2} \le n$, by the inductive hypothesis, we have $f(k) = k$. It follows that $f(n+2) = f(2k) = f(2)f(k) = 2k = n+2$. From the inequality

$$n = f(n) < f(n+1) < f(n+2) = n+2$$

we conclude that $f(n+1) = n+1$. By induction, $f(n) = n$ holds for all positive integers n. □

Second Solution. As in the previous solution, we get $f(1) = 1$. We find that

$$f(2n) = f(2)f(n) = 2f(n)$$

for all positive integers n. This implies that, for all positive integers k,

$$f\left(2^k\right) = 2^k$$

Let $k \in \mathbb{N}$. From the assumption, we obtain the inequality

$$2^k = f\left(2^k\right) < f\left(2^k + 1\right) < \cdots < f\left(2^{k+1} - 1\right) < f\left(2^{k+1}\right) = 2^{k+1}.$$

In other words, the increasing sequence of $2^k + 1$ positive integers

$$f\left(2^k\right), f\left(2^k + 1\right), \cdots, f\left(2^{k+1} - 1\right), f\left(2^{k+1}\right)$$

lies in the set of $2^k + 1$ consecutive integers $\{2^k, 2^k + 1, \cdots, 2^{k+1} - 1, 2^{k+1}\}$. This means that $f(n) = n$ for all $2^k \le n \le 2^{k+1}$. Since this holds for all positive integers k, we conclude that $f(n) = n$ for all $n \ge 2$. □

The conditions in the problem are too restrictive. Let's throw out the condition $f(2) = 2$.

Epsilon 4. *Let $f : \mathbb{N} \to \mathbb{R}^+$ be a function satisfying the conditions:*
(a) $f(mn) = f(m)f(n)$ for all positive integers m and n, and
(b) $f(n+1) \ge f(n)$ for all positive integers n.
Then, there is a constant $\alpha \in \mathbb{R}$ such that $f(n) = n^\alpha$ for all $n \in \mathbb{N}$.

We can weaken the assumption that f is completely multiplicative, but we bring back the condition $f(2) = 2$.

Epsilon 5. (Putnam 1963/A2) Let $f : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function satisfying that $f(2) = 2$ and $f(mn) = f(m)f(n)$ for all relatively prime m and n. Then, f is the identity function on $\mathbb{N}$.

In fact, we can completely drop the constraint $f(2) = 2$. In 1946, P. Erdős proved the following result in [PE]:

Theorem 1.4. *Let $f : \mathbb{N} \to \mathbb{R}$ be a function satisfying the conditions:*
(a) $f(mn) = f(m) + f(n)$ for all relatively prime m and n, and
(b) $f(n+1) \geq f(n)$ for all positive integers n.
Then, there exists a constant $\alpha \in \mathbb{R}$ such that $f(n) = \alpha \ln n$ for all $n \in \mathbb{N}$.

This implies the following multiplicative result.

Theorem 1.5. *Let $f : \mathbb{N} \to \mathbb{R}^+$ be a function satisfying the conditions:*
(a) $f(mn) = f(m)f(n)$ for all relatively prime m and n, and
(b) $f(n+1) \geq f(n)$ for all positive integers n.
Then, there is a constant $\alpha \in \mathbb{R}$ such that $f(n) = n^\alpha$ for all $n \in \mathbb{N}$.

Proof. [2] It is enough to show that the function f is completely multiplicative: $f(mn) = f(m)f(n)$ for all m and n. We split the proof in three steps.

Step 1. Let $a \geq 2$ be a positive integer and let $\Omega_a = \{x \in \mathbb{N} \mid \gcd(x, a) = 1\}$. Then, we find that

$$L := \inf_{x \in \Omega_a} \frac{f(x+a)}{f(x)} = 1$$

and

$$f\left(a^{k+1}\right) \leq f\left(a^k\right) f(a)$$

for all positive integers k.

Proof of Step 1. Since f is monotone increasing, it is clear that $L \geq 1$. Now, we notice that $f(k+a) \geq Lf(k)$ whenever $k \in \Omega_a$. Let m be a positive integer. We take a sufficiently large integer $x_0 > ma$ with $\gcd(x_0, a) = \gcd(x_0, 2) = 1$ to obtain

$$f(2)f(x_0) = f(2x_0) \geq f(x_0 + ma) \geq Lf(x_0 + (m-1)a) \geq \cdots \geq L^m f(x_0)$$

or

$$f(2) \geq L^m.$$

Since m is arbitrary, this and $L \geq 1$ force to $L = 1$. Whenever $x \in \Omega_a$, we obtain

$$\frac{f\left(a^{k+1}\right) f(x)}{f\left(a^k\right)} = \frac{f\left(a^{k+1} x\right)}{f\left(a^k\right)} \leq \frac{f\left(a^{k+1} x + a^k\right)}{f\left(a^k\right)} = f(ax+1) \leq f\left(ax + a^2\right)$$

or

$$\frac{f\left(a^{k+1}\right) f(x)}{f\left(a^k\right)} \leq f(a)f(x+a)$$

or

$$\frac{f(x+a)}{f(x)} \geq \frac{f\left(a^{k+1}\right)}{f(a)f\left(a^k\right)}.$$

[2] We present a slightly modified proof in [EH]. For another short proof, see [MJ].

It follows that

$$1=\inf_{x\in\Omega_a}\frac{f(x+a)}{f(x)}\geq\frac{f\left(a^{k+1}\right)}{f(a)f\left(a^k\right)}$$

so that

$$f\left(a^{k+1}\right)\leq f\left(a^k\right)f(a),$$

as desired.

Step 2. Similarly, we have

$$U:=\sup_{x\in\Omega_a}\frac{f(x)}{f(x+a)}=1$$

and

$$f\left(a^{k+1}\right)\geq f\left(a^k\right)f(a)$$

for all positive integers k.

Proof of Step 2. The first result immediately follows from Step 1.

$$\sup_{x\in\Omega_a}\frac{f(x)}{f(x+a)}=\frac{1}{\inf_{x\in\Omega_a}\frac{f(x+a)}{f(x)}}=1.$$

Whenever $x\in\Omega_a$ and $x>a$, we have

$$\frac{f\left(a^{k+1}\right)f(x)}{f\left(a^k\right)}=\frac{f\left(a^{k+1}x\right)}{f\left(a^k\right)}\geq\frac{f\left(a^{k+1}x-a^k\right)}{f\left(a^k\right)}=f\left(ax-1\right)\geq f\left(ax-a^2\right)$$

or

$$\frac{f\left(a^{k+1}\right)f(x)}{f\left(a^k\right)}\geq f(a)f(x-a).$$

It therefore follows that

$$1=\sup_{x\in\Omega_a}\frac{f(x)}{f(x+a)}=\sup_{x\in\Omega_a,\,x>a}\frac{f(x-a)}{f(x)}\leq\frac{f\left(a^{k+1}\right)}{f(a)f\left(a^k\right)},$$

as desired.

Step 3. From the two previous results, whenever $a\geq 2$, we have

$$f\left(a^{k+1}\right)=f\left(a^k\right)f(a).$$

Then, the straightforward induction gives that

$$f\left(a^k\right)=f(a)^k$$

for all positive integers a and k. Since f is multiplicative, whenever

$$n={p_1}^{k_1}\cdots{p_l}^{k_l}$$

gives the standard factorization of n, we obtain

$$f(n)=f\left({p_1}^{k_1}\right)\cdots f\left({p_l}^{k_l}\right)=f\left(p_1\right)^{k_1}\cdots f\left(p_l\right)^{k_l}.$$

We therefore conclude that f is completely multiplicative. $\square$

1.4. **There are Infinitely Many Primes.** The purpose of this subsection is to offer various proofs of Euclid's Theorem.

Theorem 1.6. (Euclid's Theorem) The number of primes is infinite.

Proof. Assume to the contrary $\{p_1 = 2, p_2 = 3, \cdots, p_n\}$ is the set of all primes. Consider the positive integer

$$P = p_1 \cdots p_n + 1.$$

Since $P > 1$, P must admit a prime divisor p_i for some $i \in \{1, \cdots, n\}$. Since both P and $p_1 \cdots p_n$ are divisible by p_i, we find that $1 = P - p_1 \cdots p_n$ is also divisible by p_i, which is a contradiction. $\square$

In fact, more is true. We now present four proofs of Euler's Theorem that the sum of the reciprocals of all prime numbers diverges.

Theorem 1.7. (Euler's Theorem, PEN E24) Let p_n denote the nth prime number. The infinite series

$$\sum_{n=1}^{\infty} \frac{1}{p_n}$$

diverges.

First Proof. [NZM, pp.21-23] We first prepare a lemma. Let $\varrho(n)$ denote the set of prime divisors of n. Let $\mathcal{S}_n(N)$ denote the set of positive integers $i \leq N$ satisfying that $\varrho(i) \subset \{p_1, \cdots, p_n\}$.

Lemma 1.3. *We have* $|\mathcal{S}_n(N)| \leq 2^n \sqrt{N}$.

Proof of Lemma. It is because every positive integer $i \in \mathcal{S}_n(N)$ has a unique factorization $i = st^2$, where s is a divisor of $p_1 \cdots p_n$ and $t \leq \sqrt{N}$. In other words, $i \mapsto (s, t)$ is an injective map from $\mathcal{S}_n(N)$ to $\mathcal{T}_n(N) = \{ (s,t) \mid s \,|\, p_1 \cdots p_n,\, t \leq \sqrt{N}\}$, which means that $|\mathcal{S}_n(N)| \leq |\mathcal{T}_n(N)| \leq 2^n \sqrt{N}$.

Now, assume to the contrary that the infinite series $\frac{1}{p_1} + \frac{1}{p_2} + \cdots$ converges. Then we can take a sufficiently large positive integer n satisfying that

$$\frac{1}{2} \geq \sum_{i>n}^{\infty} \frac{1}{p_i} = \frac{1}{p_{n+1}} + \frac{1}{p_{n+2}} + \cdots .$$

Take a sufficiently large positive integer N so that $N > 4^{n+1}$. By its definition of $\mathcal{S}_n(N)$, we see that each element i in $\{1, \cdots, N\} - \mathcal{S}_n(N)$ is divisible by at least one prime p_j for some $j > n$. Since the number of multiples of p_j not exceeding N is $\left\lfloor \frac{N}{p_j} \right\rfloor$, we have

$$|\{1, \cdots, N\} - \mathcal{S}_n(N)| \leq \sum_{j>n} \left\lfloor \frac{N}{p_j} \right\rfloor$$

or

$$N - |\mathcal{S}_n(N)| \leq \sum_{j>n} \left\lfloor \frac{N}{p_j} \right\rfloor \leq \sum_{j>n} \frac{N}{p_j} \leq \frac{N}{2}$$

or

$$\frac{N}{2} \leq |\mathcal{S}_n(N)| .$$

It follows from this and from the lemma that $\frac{N}{2} \leq 2^n\sqrt{N}$ so that $N \leq 4^{n+1}$. However, it is a contradiction for the choice of N. $\square$

Second Proof. We employ an auxiliary inequality without a proof.

Lemma 1.4. *The inequality $1 + t \leq e^t$ holds for all $t \in \mathbb{R}$.*

Let $n > 1$. Since each positive integer $i \leq n$ has a unique factorization $i = st^2$, where s is square free and $t \leq \sqrt{n}$, we obtain

$$\sum_{k=1}^{n} \frac{1}{k} \leq \prod_{\substack{p:prime,\\ p\leq n}} \left(1 + \frac{1}{p}\right) \sum_{t\leq\sqrt{n}} \frac{1}{t^2}.$$

Together with the estimation

$$\sum_{t=1}^{\infty} \frac{1}{t^2} \leq 1 + \sum_{t=2}^{\infty} \frac{1}{t(t-1)} = 1 + \sum_{t=2}^{\infty} \left(\frac{1}{t-1} - \frac{1}{t}\right) = 2,$$

we conclude that

$$\sum_{k=1}^{n} \frac{1}{k} \leq 2 \prod_{\substack{p:prime,\\ p\leq n}} \left(1 + \frac{1}{p}\right) \leq 2 \prod_{\substack{p:prime\\ p\leq n}} e^{\frac{1}{p}}$$

or

$$\sum_{\substack{p:prime\\ p\leq n}} \frac{1}{p} \geq \ln\left(\frac{1}{2}\sum_{k=1}^{n} \frac{1}{k}\right).$$

Since the divergence of the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \cdots$ is well-known, by Comparison Test, the series diverges. $\square$

Third Proof. [NZM, pp.21-23] We exploit an auxiliary inequality without a proof.

Lemma 1.5. *The inequality $\frac{1}{1-t} \leq e^{t+t^2}$ holds for all $t \leq \left[0, \frac{1}{2}\right]$.*

Let $l \in \mathbb{N}$. By The Fundamental Theorem of Arithmetic, each positive integer $i \leq p_l$ has a unique factorization $i = {p_1}^{e_1} \cdots {p_1}^{e_l}$ for some $e_1, \cdots, e_l \in \mathbb{Z}_{\geq 0}$. It follows that

$$\sum_{i=1}^{p_l} \frac{1}{i} \leq \sum_{e_1,\cdots,e_l\in\mathbb{Z}_{\geq 0}} \frac{1}{{p_1}^{e_1}\cdots {p_l}^{e_l}} = \prod_{j=1}^{l} \left(\sum_{k=0}^{\infty} \frac{1}{{p_j}^k}\right) = \prod_{j=1}^{l} \frac{1}{1-\frac{1}{p_j}} \leq \prod_{j=1}^{l} e^{\frac{1}{p_j}+\frac{1}{{p_j}^2}}$$

so that

$$\sum_{j=1}^{l} \left(\frac{1}{p_j} + \frac{1}{{p_j}^2}\right) \geq \ln\left(\sum_{i=1}^{p_l} \frac{1}{i}\right).$$

Together with the estimation

$$\sum_{j=1}^{\infty} \frac{1}{{p_j}^2} \leq \sum_{j=1}^{l} \frac{1}{(j+1)^2} \leq \sum_{j=1}^{l} \frac{1}{(j+1)j} = \sum_{j=1}^{l} \left(\frac{1}{j} - \frac{1}{j+1}\right) = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1,$$

we conclude that

$$\sum_{j=1}^{l} \frac{1}{p_j} \geq \ln\left(\sum_{i=1}^{p_l} \frac{1}{i}\right) - 1.$$

Since the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \cdots$ diverges, by The Comparison Test, we get the result. $\square$

Fourth Proof. [DB, p.334] It is a consequence of The Prime Number Theorem. Let $\pi(x)$ denote the prime counting function. Since The Prime Number Theorem says that $\pi(x) \to \frac{x}{\ln x}$ as $x \to \infty$, we can find a constant $\lambda > 0$ satisfying that $\pi(x) > \lambda \frac{x}{\ln x}$ for all sufficiently large positive real numbers x. This means that $n > \lambda \frac{p_n}{\ln p_n}$ when n is sufficiently large. Since $\lambda \frac{x}{\ln x} > \sqrt{x}$ for all sufficiently large $x > 0$, we also have

$$n > \lambda \frac{p_n}{\ln p_n} > \sqrt{p_n}$$

or

$$n^2 > p_n$$

for all sufficiently large n. We conclude that, when n is sufficiently large,

$$n > \lambda \frac{p_n}{\ln p_n} > \lambda \frac{p_n}{\ln (n^2)},$$

or equivalently,

$$\frac{1}{p_n} > \frac{\lambda}{2n \ln n}.$$

Since we have $\sum_{n=2}^{\infty} \frac{1}{n \ln n} = \infty$, The Comparison Test yields the desired result. $\square$

We close this subsection with a striking result establish by Viggo Brun.

Theorem 1.8. (Brun's Theorem) The sum of the reciprocals of the twin primes converges:

$$\mathcal{B} = \sum_{\text{p, p+2: prime}} \left(\frac{1}{p} + \frac{1}{p+2} \right) = \left(\frac{1}{3} + \frac{1}{5} \right) + \left(\frac{1}{5} + \frac{1}{7} \right) + \left(\frac{1}{11} + \frac{1}{13} \right) + \cdots < \infty$$

The constant $\mathcal{B} = 1.90216 \cdots$ is called Brun's Constant.

1.5. **Towards $1 Million Prize Inequalities.** In this section, we follow [JL]. We consider two conjectures.

Open Problem 1.1. (J. C. Lagarias) Given a positive integer n, let $\mathcal{H}_n$ denote the n-th harmonic number

$$\mathcal{H}_n = \sum_{i=1}^{n} \frac{1}{i} = 1 + \cdots + \frac{1}{n}$$

and let $\sigma(n)$ denote the sum of positive divisors of n. Prove that that the inequality

$$\sigma(n) \leq \mathcal{H}_n + e^{\mathcal{H}_n} \ln \mathcal{H}_n$$

holds for all positive integers n.

Open Problem 1.2. *Let π denote the prime counting function, that is, $\pi(x)$ counts the number of primes p with $1 < p \leq x$. Let $\varepsilon > 0$. Prove that that there exists a positive constant C_ε such that the inequality*

$$\left| \pi(x) - \int_2^x \frac{1}{\ln t} dt \right| \leq C_\varepsilon x^{\frac{1}{2}+\varepsilon}$$

holds for all real numbers $x \geq 2$.

These two unseemingly problems are, in fact, equivalent. Furthermore, more strikingly, they are equivalent to The Riemann Hypothesis from complex analysis. In 2000, The Clay Mathematics Institute of Cambridge, Massachusetts (CMI) has named seven prize problems. If you knock them down, you earn at least $1 Million.[3] For more info, visit the CMI website at

http://www.claymath.org/millennium

Wir müssen wissen. Wir werden wissen.

- D. Hilbert

[3]However, in that case, be aware that Mafias can knock you down and take money from you :]

2. Symmetries

Each problem that I solved became a rule, which served afterwards to solve other problems.

- R. Descartes

2.1. **Exploiting Symmetry.** We begin with the following example.

Example 4. *Let a, b, c be positive real numbers. Prove the inequality*

$$\frac{a^4+b^4}{a+b}+\frac{b^4+c^4}{b+c}+\frac{c^4+a^4}{c+a} \geq a^3+b^3+c^3.$$

First Solution. After brute-force computation, i.e, clearing denominators, we reach

$$a^5b+a^5c+b^5c+b^5a+c^5a+c^5b \geq a^3b^2c+a^3bc^2+b^3c^2a+b^3ca^2+c^3a^2b+c^3ab^2.$$

Now, we deduce

$$\begin{aligned} & a^5b+a^5c+b^5c+b^5a+c^5a+c^5b \\ = \;& a\left(b^5+c^5\right)+b\left(c^5+a^5\right)+c\left(a^5+b^5\right) \\ \geq \;& a\left(b^3c^2+b^2c^3\right)+b\left(c^3a^2+c^2b^3\right)+c\left(c^3a^2+c^2b^3\right) \\ = \;& a^3b^2c+a^3bc^2+b^3c^2a+b^3ca^2+c^3a^2b+c^3ab^2. \end{aligned}$$

Here, we used the the auxiliary inequality

$$x^5+y^5 \geq x^3y^2+x^2y^3,$$

where $x, y \geq 0$. Indeed, we obtain the equality

$$x^5+y^5-x^3y^2-x^2y^3=\left(x^3-y^3\right)\left(x^2-y^2\right).$$

It is clear that the final term $\left(x^3-y^3\right)\left(x^2-y^2\right)$ is always non-negative. □

Here goes a more economical solution without the brute-force computation.

Second Solution. The trick is to observe that the right hand side admits a nice decomposition:

$$a^3+b^3+c^3=\frac{a^3+b^3}{2}+\frac{b^3+c^3}{2}+\frac{c^3+a^3}{2}.$$

We then see that the inequality has the symmetric face:

$$\frac{a^4+b^4}{a+b}+\frac{b^4+c^4}{b+c}+\frac{c^4+a^4}{c+a} \geq \frac{a^3+b^3}{2}+\frac{b^3+c^3}{2}+\frac{c^3+a^3}{2}.$$

Now, the symmetry of this expression gives the right approach. We check that, for $x, y > 0$,

$$\frac{x^4+y^4}{x+y} \geq \frac{x^3+y^3}{2}.$$

However, we obtain the identity

$$2\left(x^4+y^4\right)-\left(x^3+y^3\right)(x+y)=x^4+y^4-x^3y-xy^3=\left(x^3-y^3\right)(x-y).$$

It is clear that the final term $\left(x^3-y^3\right)(x-y)$ is always non-negative. □

Delta 9. [LL 1967 POL] *Prove that, for all $a, b, c > 0$,*

$$\frac{a^8+b^8+c^8}{a^3b^3c^3} \geq \frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$

Delta 10. [LL 1970 AUT] *Prove that, for all $a, b, c > 0$,*

$$\frac{a+b+c}{2} \geq \frac{bc}{b+c}+\frac{ca}{c+a}+\frac{ab}{a+b}$$

Delta 11. [SL 1995 UKR] *Let n be an integer, $n \geq 3$. Let $a_1, \cdots, a_n$ be real numbers such that $2 \leq a_i \leq 3$ for $i = 1, \cdots, n$. If $s = a_1 + \cdots + a_n$, prove that*

$$\frac{{a_1}^2 + {a_2}^2 - {a_3}^2}{a_1 + a_2 + a_3} + \frac{{a_2}^2 + {a_3}^2 - {a_4}^2}{a_2 + a_3 + a_4} + \cdots + \frac{{a_n}^2 + {a_1}^2 - {a_2}^2}{a_n + a_1 + a_2} \leq 2s - 2n.$$

Delta 12. [SL 2006] *Let $a_1, \cdots, a_n$ be positive real numbers. Prove the inequality*

$$\frac{n}{2(a_1 + a_2 + \cdots + a_n)} \sum_{1 \leq i < j \leq n} a_i a_j \geq \sum_{1 \leq i < j \leq n} \frac{a_i a_j}{a_i + a_j}$$

Epsilon 6. *Let a, b, c be positive real numbers. Prove the inequality*

$$\left(1 + a^2\right)\left(1 + b^2\right)\left(1 + c^2\right) \geq (a + b)(b + c)(c + a).$$

Show that the equality holds if and only if $(a, b, c) = (1, 1, 1)$.

Epsilon 7. (Poland 2006) Let a, b, c be positive real numbers with $ab+bc+ca = abc$. Prove that

$$\frac{a^4 + b^4}{ab(a^3 + b^3)} + \frac{b^4 + c^4}{bc(b^3 + c^3)} + \frac{c^4 + a^4}{ca(c^3 + a^3)} \geq 1.$$

Epsilon 8. (APMO 1996) Let a, b, c be the lengths of the sides of a triangle. Prove that

$$\sqrt{a + b - c} + \sqrt{b + c - a} + \sqrt{c + a - b} \leq \sqrt{a} + \sqrt{b} + \sqrt{c}.$$

2.2. **Breaking Symmetry.** We now learn how to break the symmetry. Let's attack the following problem.

Example 5. *Let a, b, c be non-negative real numbers. Show the inequality*

$$a^4 + b^4 + c^4 + 3(abc)^{\frac{4}{3}} \geq 2\left(a^2b^2 + b^2c^2 + c^2a^2\right).$$

There are many ways to prove this inequality. In fact, it can be proved either with Schur's Inequality or with Popoviciu's Inequality. Here, we try to give another proof. One natural starting point is to apply The AM-GM Inequality to obtain the estimations

$$c^4 + 3(abc)^{\frac{4}{3}} \geq 4\left(c^4 \cdot (abc)^{\frac{4}{3}} \cdot (abc)^{\frac{4}{3}} \cdot (abc)^{\frac{4}{3}}\right)^{\frac{1}{4}} = 4abc^2$$

and

$$a^4 + b^4 \geq 2a^2b^2.$$

Adding these two inequalities, we obtain

$$a^4 + b^4 + c^4 + 3(abc)^{\frac{4}{3}} \geq 2a^2b^2 + 4abc^2.$$

Hence, it now remains to show that

$$2a^2b^2 + 4abc^2 \geq 2\left(a^2b^2 + b^2c^2 + c^2a^2\right)$$

or equivalently

$$0 \geq 2c^2(a-b)^2,$$

which is clearly untrue in general. **It is reversed!** However, we can exploit the above idea to finsh the proof.

Proof. Using the symmetry of the inequality, we break the symmetry. Since the inequality is symmetric, we may consider the case $a, b \geq c$ only. Since The AM-GM Inequality implies the inequality $c^4 + 3(abc)^{\frac{4}{3}} \geq 4abc^2$, we obtain the estimation

$$\begin{aligned}
& a^4 + b^4 + c^4 + 3(abc)^{\frac{4}{3}} - 2\left(a^2b^2 + b^2c^2 + c^2a^2\right) \\
\geq\ & \left(a^4 + b^4 - 2a^2b^2\right) + 4abc^2 - 2\left(b^2c^2 + c^2a^2\right) \\
=\ & \left(a^2 - b^2\right)^2 - 2c^2(a-b)^2 \\
=\ & (a-b)^2\left((a+b)^2 - 2c^2\right).
\end{aligned}$$

Since we have $a, b \geq c$, the last term is clearly non-negative. □

Epsilon 9. *Let a, b, c be the lengths of a triangle. Show that*

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} < 2.$$

Epsilon 10. (USA 1980) Prove that, for all real numbers $a, b, c \in [0, 1]$,

$$\frac{a}{b+c+1} + \frac{b}{c+a+1} + \frac{c}{a+b+1} + (1-a)(1-b)(1-c) \leq 1.$$

Epsilon 11. [AE, p. 186] Show that, for all $a, b, c \in [0, 1]$,

$$\frac{a}{1+bc} + \frac{b}{1+ca} + \frac{c}{1+ab} \leq 2.$$

Epsilon 12. [SL 2006 KOR] *Let a, b, c be the lengths of the sides of a triangle. Prove the inequality*

$$\frac{\sqrt{b+c-a}}{\sqrt{b}+\sqrt{c}-\sqrt{a}} + \frac{\sqrt{c+a-b}}{\sqrt{c}+\sqrt{a}-\sqrt{b}} + \frac{\sqrt{a+b-c}}{\sqrt{a}+\sqrt{b}-\sqrt{c}} \leq 3.$$

Epsilon 13. *Let $f(x, y) = xy\left(x^3 + y^3\right)$ for $x, y \geq 0$ with $x + y = 2$. Prove the inequality*

$$f(x, y) \leq f\left(1 + \frac{1}{\sqrt{3}}, 1 - \frac{1}{\sqrt{3}}\right) = f\left(1 - \frac{1}{\sqrt{3}}, 1 + \frac{1}{\sqrt{3}}\right).$$

Epsilon 14. *Let $a, b \geq 0$ with $a + b = 1$. Prove that*

$$\sqrt{a^2+b} + \sqrt{a+b^2} + \sqrt{1+ab} \leq 3.$$

Show that the equality holds if and only if $(a, b) = (1, 0)$ or $(a, b) = (0, 1)$.

Epsilon 15. (USA 1981) Let ABC be a triangle. Prove that

$$\sin 3A + \sin 3B + \sin 3C \leq \frac{3\sqrt{3}}{2}.$$

The above examples say that, in general, **symmetric problems does not admit symmetric solutions.** We now introduce an extremely useful inequality when we make the ordering assmption.

Epsilon 16. (Chebyshev's Inequality) Let $x_1, \cdots, x_n$ and $y_1, \cdots y_n$ be two monotone increasing sequences of real numbers:

$$x_1 \leq \cdots \leq x_n,\ y_1 \leq \cdots \leq y_n.$$

Then, we have the estimation

$$\sum_{i=1}^{n} x_i y_i \geq \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right).$$

Corollary 2.1. (The AM-HM Inequality) Let $x_1, \cdots, x_n > 0$. Then, we have

$$\frac{x_1 + \cdots + x_n}{n} \geq \frac{n}{\frac{1}{x_1} + \cdots \frac{1}{x_n}}$$

or

$$\frac{1}{x_1} + \cdots \frac{1}{x_n} \geq \frac{n^2}{x_1 + \cdots + x_n}.$$

The equality holds if and only if $x_1 = \cdots = x_n$.

Proof. Since the inequality is symmetric, we may assume that $x_1 \leq \cdots \leq x_n$. We have

$$-\frac{1}{x_1} \leq \cdots \leq -\frac{1}{x_n}.$$

Chebyshev's Inequality shows that

$$x_1 \cdot \left(-\frac{1}{x_1}\right) + \cdots + x_1 \cdot \left(-\frac{1}{x_1}\right) \geq \frac{1}{n}\left(x_1 + \cdots + x_n\right)\left[\left(-\frac{1}{x_1}\right) + \cdots + \left(-\frac{1}{x_1}\right)\right].$$

□

Remark 2.1. *In Chebyshev's Inequality, we do not require that the variables are positive. It also implies that if $x_1 \leq \cdots \leq x_n$ and $y_1 \geq \cdots \geq y_n$, then we have the reverse estimation*

$$\sum_{i=1}^{n} x_i y_i \leq \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right).$$

Epsilon 17. (United Kingdom 2002) For all $a, b, c \in (0, 1)$, show that

$$\frac{a}{1-a} + \frac{b}{1-b} + \frac{c}{1-c} \geq \frac{3\sqrt[3]{abc}}{1-\sqrt[3]{abc}}.$$

Epsilon 18. [IMO 1995/2 RUS] *Let a, b, c be positive numbers such that $abc = 1$. Prove that*

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

Epsilon 19. (Iran 1996) Let x, y, z be positive real numbers. Prove that

$$(xy + yz + zx)\left(\frac{1}{(x+y)^2} + \frac{1}{(y+z)^2} + \frac{1}{(z+x)^2}\right) \geq \frac{9}{4}.$$

We now present three different proofs of Nesbitt's Inequality:

Proposition 2.1. (Nesbitt) For all positive real numbers a, b, c, we have

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}.$$

Proof 1. *We denote $\mathcal{L}$ the left hand side. Since the inequality is symmetric in the three variables, we may assume that $a\geq b\geq c$. Since $\frac{1}{b+c}\geq\frac{1}{c+a}\geq\frac{1}{a+b}$, Chebyshev's Inequality yields that*

$$\begin{aligned}\mathcal{L} &\geq \frac{1}{3}(a+b+c)\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right)\\ &= \frac{1}{3}\left(\frac{a+b+c}{b+c}+\frac{a+b+c}{c+a}+\frac{a+b+c}{a+b}\right)\\ &= 3\left(1+\frac{a}{b+c}+1+\frac{b}{c+a}+1+\frac{c}{a+b}\right)\\ &= \frac{1}{3}(3+\mathcal{L}),\end{aligned}$$

so that $\mathcal{L}\geq\frac{3}{2}$, as desired.

Proof 2. *We now break the symmetry by a suitable normalization. Since the inequality is symmetric in the three variables, we may assume that $a\geq b\geq c$. After the substitution $x=\frac{a}{c}, y=\frac{b}{c}$, we have $x\geq y\geq 1$. It becomes*

$$\frac{\frac{a}{c}}{\frac{b}{c}+1}+\frac{\frac{b}{c}}{\frac{a}{c}+1}+\frac{1}{\frac{a}{c}+\frac{b}{c}}\geq\frac{3}{2}$$

or

$$\frac{x}{y+1}+\frac{y}{x+1}\geq\frac{3}{2}-\frac{1}{x+y}.$$

We first apply The AM-GM Inequality to deduce

$$\frac{x+1}{y+1}+\frac{y+1}{x+1}\geq 2$$

or

$$\frac{x}{y+1}+\frac{y}{x+1}\geq 2-\frac{1}{y+1}-\frac{1}{x+1}.$$

It is now enough to show that

$$2-\frac{1}{y+1}-\frac{1}{x+1}\geq\frac{3}{2}-\frac{1}{x+y}$$

or

$$\frac{1}{2}-\frac{1}{y+1}\geq\frac{1}{x+1}-\frac{1}{x+y}$$

or

$$\frac{y-1}{2(1+y)}\geq\frac{y-1}{(x+1)(x+y)}.$$

However, the last inequality clearly holds for $x\geq y\geq 1$.

Proof 3. *As in the previous proof, we may assume $a\geq b\geq 1=c$. We present a proof of*

$$\frac{a}{b+1}+\frac{b}{a+1}+\frac{1}{a+b}\geq\frac{3}{2}.$$

Let $A=a+b$ and $B=ab$. What we want to prove is

$$\frac{a^2+b^2+a+b}{(a+1)(b+1)}+\frac{1}{a+b}\geq\frac{3}{2}$$

or

$$\frac{A^2-2B+A}{A+B+1}+\frac{1}{A}\geq\frac{3}{2}$$

or

$$2A^3 - A^2 - A + 2 \geq B(7A - 2).$$

Since $7A - 2 > 2(a + b - 1) > 0$ *and* $A^2 = (a + b)^2 \geq 4ab = 4B$, *it's enough to show that*

$$4(2A^3 - A^2 - A + 2) \geq A^2(7A - 2) \quad \Leftrightarrow \quad A^3 - 2A^2 - 4A + 8 \geq 0.$$

However, it's easy to check that $A^3 - 2A^2 - 4A + 8 = (A - 2)^2(A + 2) \geq 0$.

2.3. **Symmetrizations.** We now attack non-symmetrical inequalities by transforming them into symmetric ones.

Example 6. *Let x, y, z be positive real numbers. Show the cyclic inequality*

$$\frac{x^2}{y^2}+\frac{y^2}{z^2}+\frac{z^2}{x^2}\geq\frac{x}{y}+\frac{y}{z}+\frac{z}{x}.$$

First Solution. We break the homogeneity. After the substitution $a=\frac{x}{y}, b=\frac{y}{z}, c=\frac{z}{x}$, it becomes

$$a^2+b^2+c^2\geq a+b+c.$$

We now obtain

$$a^2+b^2+c^2\geq\frac{1}{3}(a+b+c)^2\geq(a+b+c)(abc)^{\frac{1}{3}}=a+b+c.$$

□

Epsilon 20. (APMO 1991) Let $a_1,\cdots,a_n,b_1,\cdots,b_n$ be positive real numbers such that $a_1+\cdots+a_n=b_1+\cdots+b_n$. Show that

$$\frac{{a_1}^2}{a_1+b_1}+\cdots+\frac{{a_n}^2}{a_n+b_n}\geq\frac{a_1+\cdots+a_n}{2}.$$

Epsilon 21. *Let x, y, z be positive real numbers. Show the cyclic inequality*

$$\frac{x}{2x+y}+\frac{y}{2y+z}+\frac{z}{2z+x}\leq 1.$$

Epsilon 22. *Let x, y, z be positive real numbers with $x+y+z=3$. Show the cyclic inequality*

$$\frac{x^3}{x^2+xy+y^2}+\frac{y^3}{y^2+yz+z^2}+\frac{z^3}{z^2+zx+x^2}\geq 1.$$

Epsilon 23. [SL 1985 CAN] *Let x, y, z be positive real numbers. Show the cyclic inequality*

$$\frac{x^2}{x^2+yz}+\frac{y^2}{y^2+zx}+\frac{z^2}{z^2+xy}\leq 2.$$

Epsilon 24. [SL 1990 THA] *Let $a, b, c, d\geq 0$ with $ab+bc+cd+da=1$. show that*

$$\frac{a^3}{b+c+d}+\frac{b^3}{c+d+a}+\frac{c^3}{d+a+b}+\frac{d^3}{a+b+c}\geq\frac{1}{3}.$$

Delta 13. [SL 1998 MNG] *Let $a_1,\cdots,a_n$ be positive real numbers such that $a_1+\cdots+a_n<1$. Prove that*

$$\frac{a_1\cdots a_n\,(1-a_1-\cdots-a_n)}{(a_1+\cdots+a_n)\,(1-a_1)\cdots(1-a_n)}\leq\frac{1}{n^{n+1}}.$$

Don't just read it; fight it! Ask your own questions, look for your own examples, discover your own proofs. Is the hypothesis necessary? Is the converse true? What happens in the classical special case? What about the degenerate cases? Where does the proof use the hypothesis?

- P. Halmos, *I Want to be a Mathematician*

3. Geometric Inequalities

Geometry is the science of correct reasoning on incorrect figures.
- G. Pólya

3.1. **Triangle Inequalities.** Many inequalities are simplified by some suitable substitutions. We begin with a classical inequality in triangle geometry. What is the first[4] nontrivial geometric inequality?

Theorem 3.1. (Chapple 1746, Euler 1765) Let R and r denote the radii of the circumcircle and incircle of the triangle ABC. Then, we have $R \geq 2r$ and the equality holds if and only if ABC is equilateral.

Proof. Let $BC = a$, $CA = b$, $AB = c$, $s = \frac{a+b+c}{2}$ and $S = [ABC]$.[5] We now recall the well-known identities:

$$S = \frac{abc}{4R},\ S = rs,\ S^2 = s(s-a)(s-b)(s-c).$$

Hence, the inequality $R \geq 2r$ is equivalent to

$$\frac{abc}{4S} \geq 2\frac{S}{s}$$

or

$$abc \geq 8\frac{S^2}{s}$$

or

$$abc \geq 8(s-a)(s-b)(s-c).$$

We need to prove the following. □

Theorem 3.2. (A. Padoa) Let a, b, c be the lengths of a triangle. Then, we have

$$abc \geq 8(s-a)(s-b)(s-c)$$

or

$$abc \geq (b+c-a)(c+a-b)(a+b-c)$$

Here, the equality holds if and only if $a = b = c$.

Proof. We exploit The Ravi Substitution. Since a, b, c are the lengths of a triangle, there are positive reals x, y, z such that $a = y+z$, $b = z+x$, $c = x+y$. (Why?) Then, the inequality is $(y+z)(z+x)(x+y) \geq 8xyz$ for $x, y, z > 0$. However, we get

$$(y+z)(z+x)(x+y) - 8xyz = x(y-z)^2 + y(z-x)^2 + z(x-y)^2 \geq 0.$$

□

Does the above inequality hold for arbitrary positive reals a, b, c? Yes ! It's possible to prove the inequality without the additional condition that a, b, c are the lengths of a triangle :

Theorem 3.3. *Whenever x, y, $z > 0$, we have*

$$xyz \geq (y+z-x)(z+x-y)(x+y-z).$$

Here, the equality holds if and only if $x = y = z$.

[4] The first geometric inequality is the Triangle Inequality: $AB + BC \geq AC$

[5] In this book, $[P]$ stands for the area of the polygon P.

Proof. Since the inequality is symmetric in the variables, without loss of generality, we may assume that $x \geq y \geq z$. Then, we have $x + y > z$ and $z + x > y$. If $y + z > x$, then x, y, z are the lengths of the sides of a triangle. In this case, by the previous theorem, we get the result. Now, we may assume that $y + z \leq x$. Then, it is clear that $xyz > 0 \geq (y + z - x)(z + x - y)(x + y - z)$. □

The above inequality holds when some of x, y, z are zeros:

Theorem 3.4. *Let x, y, $z \geq 0$. Then, we have $xyz \geq (y + z - x)(z + x - y)(x + y - z)$.*

Proof. Since $x, y, z \geq 0$, we can find strictly positive sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ for which

$$\lim_{n\to\infty} x_n = x,\ \lim_{n\to\infty} y_n = y,\ \lim_{n\to\infty} z_n = z.$$

The above theorem says that

$$x_n y_n z_n \geq (y_n + z_n - x_n)(z_n + x_n - y_n)(x_n + y_n - z_n).$$

Now, taking the limits to both sides, we get the result.

□

We now notice that, when x, y, $z \geq 0$, the equality $xyz = (y+z-x)(z+x-y)(x+y-z)$ does not guarantee that $x = y = z$. In fact, for $x, y, z \geq 0$, the equality $xyz = (y + z - x)(z + x - y)(x + y - z)$ implies that

$$x = y = z \ \text{ or } \ x = y, z = 0 \ \text{ or } \ y = z, x = 0 \ \text{ or } \ z = x, y = 0.$$

(Verify this!) It's straightforward to verify the equality

$$xyz - (y+z-x)(z+x-y)(x+y-z) = x(x-y)(x-z) + y(y-z)(y-x) + z(z-x)(z-y).$$

Hence, it is a particular case of Schur's Inequality.

Epsilon 25. [IMO 2000/2 USA] *Let a, b, c be positive numbers such that $abc = 1$. Prove that*

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1.$$

Delta 14. *Let R and r denote the radii of the circumcircle and incircle of the right triangle ABC, resepectively. Show that*

$$R \geq (1 + \sqrt{2})r.$$

When does the equality hold ?

Delta 15. [LL 1988 ESP] *Let ABC be a triangle with inradius r and circumradius R. Show that*

$$\sin\frac{A}{2}\sin\frac{B}{2} + \sin\frac{B}{2}\sin\frac{C}{2} + \sin\frac{C}{2}\sin\frac{A}{2} \leq \frac{5}{8} + \frac{r}{4R}.$$

In 1965, W. J. Blundon[WJB] found the best possible inequalities of the form

$$\mathcal{A}(R, r) \leq s^2 \leq \mathcal{B}(R, r),$$

where $\mathcal{A}(x, y)$ and $\mathcal{B}(x, y)$ are real quadratic forms $\alpha x^2 + \beta xy + \gamma y^2$.

Delta 16. *Let R and r denote the radii of the circumcircle and incircle of the triangle ABC. Let s be the semiperimeter of ABC. Show that*

$$16Rr - 5r^2 \leq s^2 \leq 4R^2 + 4Rr + 3r^2.$$

Delta 17. [WJB2, RS] Let R and r denote the radii of the circumcircle and incircle of the triangle ABC. Let s be the semiperimeter of ABC. Show that

$$s \geq 2R + (3\sqrt{3} - 4)r.$$

Delta 18. *With the usual notation for a triangle, show the inequality*[6]

$$4R + r \geq \sqrt{3}s.$$

The Ravi Substitution is useful for inequalities for the lengths a, b, c of a triangle. After The Ravi Substitution, we can remove the condition that they are the lengths of the sides of a triangle.

Epsilon 26. [IMO 1983/6 USA] *Let a, b, c be the lengths of the sides of a triangle. Prove that*

$$a^2b(a-b) + b^2c(b-c) + c^2a(c-a) \geq 0.$$

Delta 19. (Darij Grinberg) Let a, b, c be the lengths of a triangle. Show the inequalities

$$a^3 + b^3 + c^3 + 3abc - 2b^2a - 2c^2b - 2a^2c \geq 0,$$

and

$$3a^2b + 3b^2c + 3c^2a - 3abc - 2b^2a - 2c^2b - 2a^2c \geq 0.$$

Delta 20. [LL 1983 UNK] *Show that if the sides a, b, c of a triangle satisfy the equation*

$$2\left(ab^2 + bc^2 + ca^2\right) = a^2b + b^2c + c^2a + 3abc$$

then the triangle is equilateral. Show also that the equation can be satisfied by positive real numbers that are not the sides of a triangle.

Delta 21. [IMO 1991/1 USS] *Prove for each triangle ABC the inequality*

$$\frac{1}{4} < \frac{IA \cdot IB \cdot IC}{l_A \cdot l_B \cdot l_C} \leq \frac{8}{27},$$

where I is the incenter and l_A, l_B, l_C are the lengths of the angle bisectors of ABC.

We now discuss Weitzenböck's Inequality and related theorems.

Epsilon 27. [IMO 1961/2 POL] (Weitzenböck's Inequality) Let a, b, c be the lengths of a triangle with area S. Show that

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}S.$$

Epsilon 28. (The Hadwiger-Finsler Inequality) For any triangle ABC with sides a, b, c and area F, the following inequality holds:

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}F + (a-b)^2 + (b-c)^2 + (c-a)^2$$

or

$$2ab + 2bc + 2ca - (a^2 + b^2 + c^2) \geq 4\sqrt{3}F.$$

Here is a simultaneous generalization of Weitzenböck's Inequality and Nesbitt's Inequality.

Epsilon 29. (Tsintsifas) Let p, q, r be positive real numbers and let a, b, c denote the sides of a triangle with area F. Then, we have

$$\frac{p}{q+r}a^2 + \frac{q}{r+p}b^2 + \frac{r}{p+q}c^2 \geq 2\sqrt{3}F.$$

Epsilon 30. (The Neuberg-Pedoe Inequality) Let a_1, b_1, c_1 denote the sides of the triangle $A_1B_1C_1$ with area F_1. Let a_2, b_2, c_2 denote the sides of the triangle $A_2B_2C_2$ with area F_2. Then, we have

$${a_1}^2({b_2}^2 + {c_2}^2 - {a_2}^2) + {b_1}^2({c_2}^2 + {a_2}^2 - {b_2}^2) + {c_1}^2({a_2}^2 + {b_2}^2 - {c_2}^2) \geq 16F_1F_2.$$

Notice that it's a generalization of Weitzenböck's Inequality. Carlitz observed that The Neuberg-Pedoe Inequality can be deduced from Aczél's Inequality.

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[6] It is equivalent to The Hadwiger-Finsler Inequality.

Epsilon 31. (Aczél's Inequality) If $a_1, \cdots, a_n, b_1, \cdots, b_n > 0$ satisfies the inequality

$${a_1}^2 \geq {a_2}^2 + \cdots + {a_n}^2 \text{ and } {b_1}^2 \geq {b_2}^2 + \cdots + {b_n}^2,$$

then the following inequality holds.

$$a_1b_1 - (a_2b_2 + \cdots + a_nb_n) \geq \sqrt{\left({a_1}^2 - ({a_2}^2 + \cdots + {a_n}^2)\right)\left({b_1}^2 - \left({b_2}^2 + \cdots + {b_n}^2\right)\right)}$$

3.2. **Conway Substitution.** As we saw earlier, transforming geometric inequalities to algebraic ones (and vice-versa), in order to solve them, may prove to be very useful. Besides the Ravi Substitution, we remind another technique, known to the authors as the Conway Substitution Theorem.

Theorem 3.5. (Conway) Let u, v, w be three reals such that the numbers $v+w$, $w+u$, $u+v$ and $vw+wu+uv$ are all nonnegative. Then, there exists a triangle XYZ with sidelengths $x = YZ = \sqrt{v+w}$, $y = ZX = \sqrt{w+u}$, $z = XY = \sqrt{u+v}$. This triangle satisfies $y^2+z^2-x^2 = 2u$, $z^2+x^2-y^2 = 2v$, $x^2+y^2-z^2 = 2w$. The area T of this triangle equals $T = \frac{1}{2}\sqrt{vw+wu+uv}$. If $X = \angle ZXY, Y = \angle XYZ, Z = \angle YZX$ are the angles of this triangle, then $\cot X = \frac{u}{2T}$, $\cot Y = \frac{v}{2T}$ and $\cot Z = \frac{w}{2T}$.

Proof. Since the numbers $v+w$, $w+u$, $u+v$ are nonnegative, their square roots $\sqrt{v+w}$, $\sqrt{w+u}$, $\sqrt{u+v}$ exist, and, of course, are nonnegative as well. A straightforward computation shows that $\sqrt{w+u}+\sqrt{u+v} \geq \sqrt{v+w}$. Similarly, $\sqrt{u+v}+\sqrt{v+w} \geq \sqrt{w+u}$ and $\sqrt{v+w}+\sqrt{w+u} \geq \sqrt{u+v}$. Thus, there exists a triangle XYZ with sidelengths

$$x = YZ = \sqrt{v+w},\ y = ZX = \sqrt{w+u},\ z = XY = \sqrt{u+v}.$$

It follows that

$$y^2+z^2-x^2 = \left(\sqrt{w+u}\right)^2 + \left(\sqrt{u+v}\right)^2 - \left(\sqrt{v+w}\right)^2 = 2u.$$

Similarly, $z^2+x^2-y^2 = 2v$ and $x^2+y^2-z^2 = 2w$. According now to the fact that

$$\cot Z = \frac{x^2+y^2-z^2}{4T},$$

we deduce that so that $\cot Z = \frac{w}{2T}$, and similarly $\cot X = \frac{u}{2T}$ and $\cot Y = \frac{v}{2T}$. The well-known trigonometric identity

$$\cot Y \cdot \cot Z + \cot Z \cdot \cot X + \cot X \cdot \cot Y = 1,$$

now becomes

$$\frac{v}{2T}\cdot\frac{w}{2T} + \frac{w}{2T}\cdot\frac{u}{2T} + \frac{u}{2T}\cdot\frac{v}{2T} = 1$$

or

$$vw+wu+uv = 4T^2.$$

or

$$T = \frac{1}{2}\sqrt{4T^2} = \frac{1}{2}\sqrt{vw+wu+uv}.$$

□

Note that the positive real numbers m, n, p satisfy the above conditions, and therefore, there exists a triangle with sidelengths $m = \sqrt{n+p}$, $n = \sqrt{p+m}$, $p = \sqrt{m+n}$. However, we will further see that there are such cases when we need the version in which the numbers m, n, p are not all necessarily nonnegative.

Delta 22. (Turkey 2006) If x, y, z are positive numbers with $xy+yz+zx = 1$, show that

$$\frac{27}{4}(x+y)(y+z)(z+x) \geq (\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x})^2 \geq 6\sqrt{3}.$$

We continue with an interesting inequality discussed on the MathLinks Forum.

Proposition 3.1. *If x, y, z are three reals such that the numbers $y+z$, $z+x$, $x+y$ and $yz+zx+xy$ are all nonnegative, then*

$$\sum \sqrt{(z+x)(x+y)} \geq x+y+z+\sqrt{3}\cdot\sqrt{yz+zx+xy}.$$

Proof. (Darij Grinberg) Applying the Conway substitution theorem to the reals x, y, z, we see that, since the numbers $y+z$, $z+x$, $x+y$ and $yz+zx+xy$ are all nonnegative, we can conclude that there exists a triangle ABC with sidelengths $a = BC = \sqrt{y+z}$, $b = CA = \sqrt{z+x}$, $c = AB = \sqrt{x+y}$ and area $S = \frac{1}{2}\sqrt{yz+zx+xy}$. Now, we have

$$\sum \sqrt{(z+x)(x+y)} = \sum \sqrt{z+x} \cdot \sqrt{x+y} = \sum b \cdot c = bc + ca + ab,$$

$$x+y+z = \frac{1}{2}\left(\left(\sqrt{y+z}\right)^2 + \left(\sqrt{z+x}\right)^2 + \left(\sqrt{x+y}\right)^2\right) = \frac{1}{2}\left(a^2+b^2+c^2\right),$$

and

$$\sqrt{3} \cdot \sqrt{yz+zx+xy} = 2\sqrt{3} \cdot \frac{1}{2}\sqrt{yz+zx+xy} = 2\sqrt{3} \cdot S.$$

Hence, the inequality in question becomes

$$bc + ca + ab \geq \frac{1}{2}\left(a^2+b^2+c^2\right) + 2\sqrt{3} \cdot S,$$

which is equivalent with

$$a^2+b^2+c^2 \geq 4\sqrt{3} \cdot S + (b-c)^2 + (c-a)^2 + (a-b)^2.$$

But this is the well-known refinement of the Weintzenbock Inequality, discovered by Finsler and Hadwiger in 1937. See [FiHa]. □

Five years later, Pedoe [DP2] proved a magnificent generalization of the same Weitzenböck Inequality. In Mitrinovic, Pecaric, and Volenecs' classic *Recent Advances in Geometric Inequalities*, this generalization is referred to as the Neuberg-Pedoe Inequality. See also [DP1], [DP2], [DP3], [DP5] and [JN].

Proposition 3.2. (Neuberg-Pedoe) Let a, b, c, and x, y, z be the side lengths of two given triangles ABC, XYZ with areas S, and T, respectively. Then,

$$a^2\left(y^2+z^2-x^2\right) + b^2\left(z^2+x^2-y^2\right) + c^2\left(x^2+y^2-z^2\right) \geq 16ST,$$

with equality if and only if the triangles ABC and XYZ are similar.

Proof. (Darij Grinberg) First note that the inequality is homogeneous in the sidelengths x, y, z of the triangle XYZ (in fact, these sidelengths occur in the power 2 on the left hand side, and on the right hand side they occur in the power 2 as well, since the area of a triangle is quadratically dependant from its sidelengths). Hence, this inequality is invariant under any similitude transformation executed on triangle XYZ. In other words, we can move, reflect, rotate and stretch the triangle XYZ as we wish, but the inequality remains equivalent. But, of course, by applying similitude transformations to triangle XYZ, we can always achieve a situation when $Y = B$ and $Z = C$ and the point X lies in the same half-plane with respect to the line BC as the point A. Hence, in order to prove the Neuberg-Pedoe Inequality for *any* two triangles ABC and XYZ, it is enough to prove it for two triangles ABC and XYZ in this special situation.

So, assume that the triangles ABC and XYZ are in this special situation, i. e. that we have $Y = B$ and $Z = C$ and the point X lies in the same half-plane with respect to the line BC as the point A. We, thus, have to prove the inequality

$$a^2\left(y^2+z^2-x^2\right) + b^2\left(z^2+x^2-y^2\right) + c^2\left(x^2+y^2-z^2\right) \geq 16ST.$$

Well, by the cosine law in triangle ABX, we have

$$AX^2 = AB^2 + XB^2 - 2 \cdot AB \cdot XB \cdot \cos\angle ABX.$$

Let's figure out now what this equation means. At first, $AB = c$. Then, since $B = Y$, we have $XB = XY = z$. Finally, we have either $\angle ABX = \angle ABC - \angle XBC$ or $\angle ABX = \angle XBC - \angle ABC$ (depending on the arrangement of the points), but in both cases $\cos\angle ABX = \cos(\angle ABC - \angle XBC)$. Since $B = Y$ and $C = Z$, we can rewrite the angle $\angle XBC$ as $\angle XYZ$. Thus,

$$\cos\angle ABX = \cos(\angle ABC - \angle XYZ) = \cos\angle ABC \cos\angle XYZ + \sin\angle ABC \sin\angle XYZ.$$

By the Cosine Law in triangles ABC and XYZ, we have

$$\cos\angle ABC = \frac{c^2+a^2-b^2}{2ca}, \quad \text{and} \quad \cos\angle XYZ = \frac{z^2+x^2-y^2}{2zx}.$$

Also, since

$$\sin\angle ABC = \frac{2S}{ca}, \quad \text{and} \quad \sin\angle XYZ = \frac{2T}{zx},$$

we have that

$$\begin{aligned} & \cos\angle ABX \\ = \ & \cos\angle ABC \cos\angle XYZ + \sin\angle ABC \sin\angle XYZ \\ = \ & \frac{c^2+a^2-b^2}{2ca} \cdot \frac{z^2+x^2-y^2}{2zx} + \frac{2S}{ca} \cdot \frac{2T}{zx}. \end{aligned}$$

This makes the equation

$$AX^2 = AB^2 + XB^2 - 2 \cdot AB \cdot XB \cdot \cos\angle ABX$$

transform into

$$AX^2 = c^2 + z^2 - 2 \cdot c \cdot z \cdot \left(\frac{c^2+a^2-b^2}{2ca} \cdot \frac{z^2+x^2-y^2}{2zx} + \frac{2S}{ca} \cdot \frac{2T}{zx} \right),$$

which immediately simplifies to

$$AX^2 = c^2 + z^2 - 2 \left(\frac{\left(c^2+a^2-b^2\right)\left(z^2+x^2-y^2\right)}{4ax} + \frac{4ST}{ax} \right),$$

and since $YZ = BC$,

$$AX^2 = \frac{\left(a^2\left(y^2+z^2-x^2\right) + b^2\left(z^2+x^2-y^2\right) + c^2\left(x^2+y^2-z^2\right)\right) - 16ST}{2ax}.$$

Thus, according to the (obvious) fact that $AX^2 \geq 0$, we conclude that

$$a^2\left(y^2+z^2-x^2\right) + b^2\left(z^2+x^2-y^2\right) + c^2\left(x^2+y^2-z^2\right) \geq 16ST,$$

which proves the Neuberg-Pedoe Inequality. The equality holds if and only if the points A and X coincide, i. e. if the triangles ABC and XYZ are congruent. Now, of course, since the triangle XYZ we are dealing with is not the initial triangle XYZ, but just its image under a similitude transformation, the general equality condition is that the triangles ABC and XYZ are similar (not necessarily being congruent). $\square$

Delta 23. (Bottema [BK]) Let a, b, c, and x, y, z be the side lengths of two given triangles ABC, XYZ with areas S, and T, respectively. If P is an arbitrary point in the plane of triangle ABC, then we have the inequality

$$x \cdot AP + y \cdot BP + z \cdot CP \geq \sqrt{\frac{a^2\left(y^2+z^2-x^2\right) + b^2\left(z^2+x^2-y^2\right) + c^2\left(x^2+y^2-z^2\right)}{2} + 8ST}.$$

Epsilon 32. *If A, B, C, X, Y, Z denote the magnitudes of the corresponding angles of triangles ABC, and XYZ, respectively, then*

$$\cot A \cot Y + \cot A \cot Z + \cot B \cot Z + \cot B \cot X + \cot C \cot X + \cot C \cot Y \geq 2.$$

Epsilon 33. (Vasile Cârtoaje) Let a, b, c, x, y, z be nonnegative reals. Prove the inequality
$$(ay + az + bz + bx + cx + cy)^2 \geq 4 (bc + ca + ab) (yz + zx + xy),$$
with equality if and only if $a : x = b : y = c : z$.

Delta 24. (The Extended Tsintsifas Inequality) *Let p, q, r be positive real numbers such that the terms $q+r$, $r+p$, $p+q$ are all positive, and let a, b, c denote the sides of a triangle with area F. Then, we have*
$$\frac{p}{q+r}a^2 + \frac{q}{r+p}b^2 + \frac{r}{p+q}c^2 \geq 2\sqrt{3}F.$$

Epsilon 34. (Walter Janous, Crux Mathematicorum) If u, v, w, x, y, z are six reals such that the terms $y+z$, $z+x$, $x+y$, $v+w$, $w+u$, $u+v$, and $vw+wu+uv$ are all nonnegative, then
$$\frac{x}{y+z} \cdot (v+w) + \frac{y}{z+x} \cdot (w+u) + \frac{z}{x+y} \cdot (u+v) \geq \sqrt{3 (vw + wu + uv)}.$$

Note that the Neuberg-Pedoe Inequality is a generalization (actually the better word is *parametrization*) of the Weitzenböck Inequality. How about deducing Hadwiger-Finsler's Inequality from it? Apparently this is not possible. However, the Conway Substitution Theorem will change our mind.

Lemma 3.1. *Let ABC be a triangle with side lengths a, b, c, and area S, and let u, v, w be three reals such that the numbers $v + w$, $w + u$, $u + v$ and $vw + wu + uv$ are all nonnegative. Then,*
$$ua^2 + vb^2 + wc^2 \geq 4\sqrt{vw + wu + uv} \cdot S.$$

Proof. According to the Conway Substitution Theorem, we can construct a triangle with sidelenghts $x = \sqrt{v+w}$, $y = \sqrt{w+u}$, $z = \sqrt{u+v}$ and area $T = \sqrt{vw+wu+uv}/2$. Let this triangle be XYZ. In this case, by the Neuberg-Pedoe Inequality, applied for the triangles ABC and XYZ, we get that
$$a^2 \left(y^2 + z^2 - x^2\right) + b^2 \left(z^2 + x^2 - y^2\right) + c^2 \left(x^2 + y^2 - z^2\right) \geq 16ST.$$
By the formulas given in the Conway Substitution Theorem, this becomes equivalent with
$$a^2 \cdot 2u + b^2 \cdot 2v + c^2 \cdot 2w \geq 16S \cdot \frac{1}{2}\sqrt{vw + wu + uv}$$
which simplifies to $ua^2 + vb^2 + wc^2 \geq 4\sqrt{vw + wu + uv} \cdot S$. □

Proposition 3.3. (Cosmin Pohoaţă) Let ABC be a triangle with side lengths a, b, c, and area S and let x, y, z be three positive real numbers. Then,
$$a^2 + b^2 + c^2 \geq 4\sqrt{3}S + \frac{2}{x+y+z}\left(\frac{x^2 - yz}{x} \cdot a^2 + \frac{y^2 - zx}{y} \cdot b^2 + \frac{z^2 - xy}{z} \cdot c^2\right).$$

Proof. Let $m = xyz(x + y + z) - 2yz(x^2 - yz)$, $n = xyz(x + y + z) - 2zx(y^2 - zx)$, and $p = xyz(x + y + z) - 2xy(z^2 - xy)$. The three terms $n + p$, $p + m$, and $m + n$ are all positive, and since
$$mn + np + pm = 3x^2y^2z^2(x + y + z)^2 \geq 0,$$
by Lemma 3.1, we get that
$$\sum_{cyc} [xyz(x + y + z) - 2yz(x^2 - yz)]a^2 \geq 4xyz(x + y + z)\sqrt{3}S.$$

This rewrites as

$$\sum_{cyc}\left[(x+y+z)-2\cdot\frac{x^2-yz}{x}\right]a^2 \geq 4(x+y+z)\sqrt{3}S,$$

and, thus,

$$a^2+b^2+c^2 \geq 4\sqrt{3}S+\frac{2}{x+y+z}\left(\frac{x^2-yz}{x}\cdot a^2+\frac{y^2-zx}{y}\cdot b^2+\frac{z^2-xy}{z}\cdot c^2\right).$$

□

Obviously, for $x=a$, $y=b$, $z=c$, and following the fact that

$$a^3+b^3+c^3-3abc=\frac{1}{2}(a+b+c)\left[(a-b)^2+(b-c)^2+(c-a)^2\right],$$

Proposition 3.3 becomes equivalent with the Hadwiger-Finsler Inequality. Note also that for $x=y=z$, Proposition 3.3 turns out to be the Weintzenbock Inequality. Therefore, by using only Conway's Substitution Theorem, we've transformed a result which strictly generalizes the Weintzebock Inequality (the Neuberg-Pedoe Inequality) into one which generalizes both the Weintzenbock Inequality and, surprisingly or not, the Hadwiger-Finsler Inequality.

3.3. **Hadwiger-Finsler Revisited.** The Hadwiger-Finsler inequality is known in literature as a refinement of Weitzenböck's Inequality. Due to its great importance and beautiful aspect, many proofs for this inequality are now known. For example, in [AE] one can find eleven proofs. Is the Hadwiger-Inequality the best we can do? The answer is indeed no. Here, we shall enlighten a few of its sharpening.

We begin with an interesting "phenomenon". Most of you might know that according to the formulas $ab + bc + ca = s^2 + r^2 + 4Rr$, and $a^2 + b^2 + c^2 = 2(s^2 - r^2 - 4Rr)$, the Hadwiger-Finsler Inequality rewrites as

$$4R + r \geq s\sqrt{3},$$

where s is the semiperimeter of the triangle. However, by using this last **equivalent** form in a trickier way, we may obtain a slightly **sharper** result:

Proposition 3.4. (Cezar Lupu, Cosmin Pohoaţă) In any triangle ABC with sidelengths a, b, c, circumradius R, inradius r, and area S, we have that

$$a^2 + b^2 + c^2 \geq 2S\sqrt{3} + 2r(4R + r) + (a - b)^2 + (b - c)^2 + (c - a)^2.$$

Proof. As announced, we start with

$$4R + r \geq s\sqrt{3}.$$

By multiplying with 2 and adding $2r(4R + r)$ to both terms, we obtain that

$$16Rr + 4r^2 \geq 2S\sqrt{3} + 2r(4R + r).$$

According now to the fact that $ab+bc+ca = s^2+r^2+4Rr$, and $a^2+b^2+c^2 = 2(s^2-r^2-4Rr)$, this rewrites as

$$2(ab + bc + ca) - (a^2 + b^2 + c^2) \geq 2S\sqrt{3} + 2r(4R + r).$$

Therefore, we obtain

$$a^2 + b^2 + c^2 \geq 2S\sqrt{3} + 2r(4R + r) + (a - b)^2 + (b - c)^2 + (c - a)^2.$$

□

This might seem strange, but wait until you see how does the geometric version of Schur's Inequality look like (of course, since we expect to run through another refinement of the Hadwiger-Finsler Inequality, we obviously refer to the third degree case of Schur's Inequality).

Proposition 3.5. (Cezar Lupu, Cosmin Pohoaţă [LuPo]) In any triangle ABC with sidelengths a, b, c, circumradius R, inradius r, and area S, we have that

$$a^2 + b^2 + c^2 \geq 4S\sqrt{3 + \frac{4(R - 2r)}{4R + r}} + (a - b)^2 + (b - c)^2 + (c - a)^2.$$

Proof. The third degree case of Schur's Inequality says that for any three nonnegative real numbers m, n, p, we have that

$$m^3 + n^3 + p^3 + 3mnp \geq m^2(n + p) + n^2(p + m) + p^2(m + n).$$

Note that this can be rewritten as

$$2(np + pm + mn) - (m^2 + n^2 + p^2) \leq \frac{9mnp}{m + n + p},$$

and by plugging in the substitutions $x = \frac{1}{m}$, $y = \frac{1}{n}$, and $z = \frac{1}{p}$, we obtain that

$$\frac{yz}{x} + \frac{zx}{y} + \frac{xy}{z} + \frac{9xyz}{yz + zx + xy} \geq 2(x + y + z).$$

So far so good, but let's take this now geometrically. Using the Ravi Substitution (i. e.

$$x = \frac{1}{2}(b+c-a), \quad y = \frac{1}{2}(c+a-b), \quad \text{and } p = \frac{1}{2}(a+b-c),$$

where a, b, c are the sidelengths of triangle ABC), we get that the above inequality rewrites as

$$\sum_{cyc} \frac{(b+c-a)(c+a-b)}{(a+b-c)} + \frac{9(b+c-a)(c+a-b)(a+b-c)}{\sum(b+c-a)(c+a-b)} \geq 2(a+b+c).$$

Since $ab+bc+ca = s^2+r^2+4Rr$ and $a^2+b^2+c^2 = 2(s^2-r^2-4Rr)$, it follows that

$$\sum_{cyc}(b+c-a)(c+a-b) = 4r(4R+r).$$

Thus, according to Heron's area formula that

$$S = \sqrt{s(s-a)(s-b)(s-c)},$$

we obtain

$$\sum \frac{(b+c-a)(c+a-b)}{(a+b-c)} + \frac{18sr}{4R+r} \geq 4s.$$

This is now equivalent to

$$\sum_{cyc} \frac{(s-a)(s-b)}{(s-c)} + \frac{9sr}{4R+r} \geq 2s,$$

and so

$$\sum_{cyc} (s-a)^2(s-b)^2 + \frac{9s^2r^3}{4R+r} \geq 2s^2r^2.$$

By the identity

$$\sum_{cyc} (s-a)^2(s-b)^2 = \left(\sum_{cyc} (s-a)(s-b)\right)^2 - 2s^2r^2,$$

we have

$$\left(\sum_{cyc} (s-a)(s-b)\right)^2 - 2s^2r^2 + \frac{9s^2r^3}{4R+r} \geq 2s^2r^2,$$

and since

$$\sum_{cyc} (s-a)(s-b) = r(4R+r),$$

we deduce that

$$r^2(4R+r)^2 + \frac{9s^2r^3}{4R+r} \geq 4s^2r^2.$$

This finally rewrites as

$$\left(\frac{4R+r}{s}\right)^2 + \frac{9r}{4R+r} \geq 4.$$

According again to $ab+bc+ca = s^2+r^2+4Rr$ and $a^2+b^2+c^2 = 2(s^2-r^2-4Rr)$, we have

$$\left(\frac{2(ab+bc+ca)-(a^2+b^2+c^2)}{4S}\right)^2 \geq 4 - \frac{9r}{4R+r}.$$

Therefore,

$$a^2+b^2+c^2 \geq 4S\sqrt{3+\frac{4(R-2r)}{4R+r}} + (a-b)^2+(b-c)^2+(c-a)^2.$$

□

Epsilon 35. (Tran Quang Hung) In any triangle ABC with sidelengths a, b, c, circumradius R, inradius r, and area S, we have

$$a^2+b^2+c^2 \geq 4S\sqrt{3}+(a-b)^2+(b-c)^2+(c-a)^2+16Rr\left(\sum\cos^2\frac{A}{2}-\sum\cos\frac{B}{2}\cos\frac{C}{2}\right).$$

Delta 25. *Let a, b, c be the lengths of a triangle with area S.*
(a) (Cosmin Pohoaţă) Prove that

$$a^2+b^2+c^2 \geq 4S\sqrt{3}+\frac{1}{2}\left(|a-b|+|b-c|+|c-a|\right)^2.$$

(b) Show that, for all positive integers n,

$$a^{2n}+b^{2n}+c^{2n} \geq 3\left(\frac{4}{\sqrt{3}}\right)^n+(a-b)^{2n}+(b-c)^{2n}+(c-a)^{2n}.$$

3.4. **Trigonometry Rocks!** Trigonometry is an extremely powerful tool in geometry. We begin with Fagnano's theorem that among all inscribed triangles in a given acute-angled triangle, the feet of its altitudes are the vertices of the one with the least perimeter. Despite of its apparent simplicity, the problem proved itself really challenging and attractive to many mathematicians of the twentieth century. Several proofs are presented at [Fag].

Theorem 3.6. (Fagnano's Theorem) Let ABC be any triangle, with sidelengths a, b, c, and area S. If XYZ is inscribed in ABC, then

$$XY + YZ + ZX \geq \frac{8S^2}{abc}.$$

Equality holds if and only if ABC is acute-angled, and then only if XYZ is its orthic triangle.

Proof. (Finbarr Holland [FH]) Let XYZ be a triangle inscribed in ABC. Let $x = BX$, $y = CY$, and $z = AZ$. Then $0 < x < a$, $0 < y < b$, $0 < z < c$. By applying the Cosine Law in the triangle ZBX, we have

$$\begin{aligned} ZX^2 &= (c-z)^2 + x^2 - 2x(c-z)\cos B \\ &= (c-z)^2 + x^2 + 2xc - z)\cos(A+C) \\ &= (x\cos A + (c-z)\cos C)^2 + (x\sin A - (c-z)\sin C)^2. \end{aligned}$$

Hence, we have

$$ZX \geq |x\cos A + (c-z)\cos C|,$$

with equality if and only if $x\sin A = (c-z)\sin C$ or $ax + cz = c^2$, Similarly, we obtain

$$XY \geq |y\cos B + (a-x)\cos A|,$$

with equality if and only if $ax + by = a^2$. And

$$YZ \geq |z\cos C + (b-y)\cos B|,$$

with equality if and only if $by + cz = b^2$. Thus, we get

$$\begin{aligned} & XY + YZ + ZX \\ \geq\ & |y\cos B + (a-x)\cos A| + |z\cos C + (b-y)\cos B| + |x\cos A + (c-z)\cos C| \\ \geq\ & |y\cos B + (a-x)\cos A + z\cos C + (b-y)\cos B + x\cos A + (c-z)\cos C| \\ \geq\ & |a\cos A + b\cos B + c\cos C| \\ =\ & \frac{a^2(b^2+c^2-a^2) + b^2(c^2+a^2-b^2) + c^2(a^2+b^2-c^2)}{2abc} \\ =\ & \frac{8S^2}{abc}. \end{aligned}$$

Note that we have equality here if and only if

$$ax + cz = c^2, \quad ax + by = a^2, \quad \text{and } by + cz = b^2,$$

and moreover the expressions

$$u = x\cos A + (c-z)\cos C, \quad v = y\cos B + (a-x)\cos A, \quad w = z\cos C + (b-y)\cos B,$$

are either all negative or all nonnegative. Now it is easy to very that the system of equations

$$ax + cz = c^2, \quad ax + by = a^2, \quad by + cz = b^2$$

has an unique solution given by

$$x = c\cos B, \quad y = a\cos C, \quad \text{and } z = b\cos A,$$

in which case

$$u = b\cos C, \quad v = c\cos C, \quad \text{and } w = a\cos A.$$

Thus, in this case, at most one of u, v, w can be negative. But, if one of u, v, w is zero, then one of x, y, z must be zero, which is not possible. It follows that

$$XY + YZ + ZX > \frac{8S^2}{abc},$$

unless ABC is acute-angled, and XYZ is its orthic triangle. If ABC is acute-angled, then $\frac{8S^2}{abc}$ is the perimeter of its orthic triangle, in which case we recover Fagnano's theorem. □

We continue with Morley's miracle. We first prepare two well-known trigonometric identities.

Epsilon 36. *For all $\theta \in \mathbb{R}$, we have*

$$\sin(3\theta) = 4\sin\theta \sin\left(\frac{\pi}{3}+\theta\right)\sin\left(\frac{2\pi}{3}+\theta\right).$$

Epsilon 37. *For all $A, B, C \in \mathbb{R}$ with $A + B + C = 2\pi$, we have*

$$\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A\cos B\cos C = 1.$$

Theorem 3.7. (Morley's Theorem) The three points of intersections of the adjacent internal angle trisectors of a triangle forms an equilateral triangle.

Proof. We want to show that the triangle $E_1E_2E_3$ is equilateral.

Let $\mathcal{R}$ denote the circumradius of $A_1A_2A_3$. Setting $\angle A_i = 3\theta_i$ for $i = 1, 2, 3$, we get $\theta_1 + \theta_2 + \theta_3 = \frac{\pi}{3}$. We now apply The Sine Law twice to deduce

$$A_1E_3 = \frac{\sin\theta_2}{\sin(\pi-\theta_1-\theta_2)}\,A_1A_2 = \frac{\sin\theta_2}{\sin\left(\frac{2\pi}{3}+\theta_3\right)}\cdot 2\mathcal{R}\sin(3\theta_3) = 8\mathcal{R}\sin\theta_2\sin\theta_3\sin\left(\frac{\pi}{3}+\theta_3\right).$$

By symmetry, we also have

$$A_1E_2 = 8\mathcal{R}\sin\theta_3\sin\theta_2\sin\left(\frac{\pi}{3}+\theta_2\right).$$

Now, we present two different ways to complete the proof. The first method is more direct and the second one gives more information.

First Method. One of the most natural approaches to crack this is to compute the lengths of $E_1E_2E_3$. We apply The Cosine Law to obtain

$$\begin{aligned}
&{E_1E_2}^2\\
=\;& {AE_3}^2 + {AE_2}^2 - 2\cos(\angle E_3A_1E_2)\cdot AE_3\cdot AE_1\\
=\;& 64\mathcal{R}^2\sin^2\theta_2\sin^2\theta_3\left[\sin^2\left(\frac{\pi}{3}+\theta_3\right)+\sin^2\left(\frac{\pi}{3}+\theta_2\right)-2\cos\theta_1\sin\left(\frac{\pi}{3}+\theta_3\right)\sin\left(\frac{\pi}{3}+\theta_2\right)\right]
\end{aligned}$$

To avoid long computation, here, we employ a trick. In the view of the equality

$$(\pi-\theta_1)+\left(\frac{\pi}{6}-\theta_2\right)+\left(\frac{\pi}{6}-\theta_3\right)=\pi,$$

we have

$$\cos^2(\pi-\theta_1)+\cos^2\left(\frac{\pi}{6}-\theta_2\right)+\cos^2\left(\frac{\pi}{6}-\theta_3\right)+2\cos(\pi-\theta_1)\cos\left(\frac{\pi}{6}-\theta_2\right)\cos\left(\frac{\pi}{6}-\theta_3\right)=1$$

or

$$\cos^2\theta_1+\sin^2\left(\frac{\pi}{3}+\theta_2\right)+\sin^2\left(\frac{\pi}{3}+\theta_3\right)-2\cos\theta_1\left(\frac{\pi}{3}+\theta_2\right)\cos\left(\frac{\pi}{3}+\theta_3\right)=1$$

or

$$\sin^2\left(\frac{\pi}{3}+\theta_2\right)+\sin^2\left(\frac{\pi}{3}+\theta_3\right)-2\cos\theta_1\sin\left(\frac{\pi}{3}+\theta_2\right)\sin\left(\frac{\pi}{3}+\theta_3\right)=\sin^2\theta_1.$$

We therefore find that

$${E_1E_2}^2 = 64\mathcal{R}^2\sin^2\theta_1\sin^2\theta_2\sin^2\theta_3$$

so that

$$E_1E_2 = 8\mathcal{R}\sin\theta_1\sin\theta_2\sin\theta_3$$

Remarkably, the length of E_1E_2 is symmetric in the angles! By symmetry, we therefore conclude that $E_1E_2E_3$ is an equilateral triangle with the length $8R\sin\theta_1\sin\theta_2\sin\theta_3$.

Second Method. We find the angles in the picture explicitly. Look at the triangle $E_3A_1E_2$. The equality

$$\theta_1 + \left(\frac{\pi}{3} + \theta_2\right) + \left(\frac{\pi}{3} + \theta_3\right) = \pi$$

allows us to invite a ghost triangle ABC having the angles

$$A = \theta_1,\ B = \frac{\pi}{3} + \theta_2,\ C = \frac{\pi}{3} + \theta_3.$$

Observe that two triangles BAC and $E_3A_1E_2$ are similar. Indeed, we have $\angle BAC = \angle E_3A_1E_2$ and

$$\frac{A_1E_3}{A_1E_2} = \frac{8\mathcal{R}\sin\theta_2\sin\theta_3\sin\left(\frac{\pi}{3}+\theta_3\right)}{8\mathcal{R}\sin\theta_3\sin\theta_2\sin\left(\frac{\pi}{3}+\theta_2\right)} = \frac{\sin\left(\frac{\pi}{3}+\theta_3\right)}{\sin\left(\frac{\pi}{3}+\theta_2\right)} = \frac{\sin C}{\sin B} = \frac{AB}{AC}.$$

It therefore follows that

$$(\angle A_1E_3E_2,\ \angle A_1E_2E_3) = \left(\frac{\pi}{3} + \theta_2,\ \frac{\pi}{3} + \theta_3\right).$$

Similarly, we also have

$$(\angle A_2E_1E_3,\ \angle A_2E_3E_1) = \left(\frac{\pi}{3} + \theta_3,\ \frac{\pi}{3} + \theta_1\right).$$

and

$$(\angle A_3E_2E_1,\ \angle A_3E_1E_2) = \left(\frac{\pi}{3} + \theta_1,\ \frac{\pi}{3} + \theta_2\right).$$

An angle computation yields

$$\begin{aligned}
\angle E_1E_2E_3 &= 2\pi - (\angle A_1E_2E_3 + \angle E_1E_2A_3 + \angle A_3E_2A_1)\\
&= 2\pi - \left[\ \left(\frac{\pi}{3}+\theta_3\right) + \left(\frac{\pi}{3}+\theta_1\right) + (\pi - \theta_3 - \theta_1)\ \right]\\
&= \frac{\pi}{3}.
\end{aligned}$$

Similarly, we also have $\angle E_2E_3E_1 = \frac{\pi}{3} = \angle E_3E_1E_2$. It follows that $E_1E_2E_3$ is equilateral. Furthermore, we apply The Sine Law to reach

$$\begin{aligned}
E_2E_3 &= \frac{\sin\theta_1}{\sin\left(\frac{\pi}{3}+\theta_3\right)}\ A_1E_3\\
&= \frac{\sin\theta_1}{\sin\left(\frac{\pi}{3}+\theta_3\right)} \cdot 8\mathcal{R}\sin\theta_2\sin\theta_3\sin\left(\frac{\pi}{3}+\theta_3\right)\\
&= 8\mathcal{R}\sin\theta_1\sin\theta_2\sin\theta_3.
\end{aligned}$$

Hence, we find that the triangle $E_1E_2E_3$ has the length $8\mathcal{R}\sin\theta_1\sin\theta_2\sin\theta_3$. □

We pass now to another 'miracle': the Steiner-Lehmus theorem.

Theorem 3.8. (The Steiner-Lehmus Theorem) If the internal angle-bisectors of two angles of a triangle are congruent, then the triangle is isosceles.

Proof. [MH] Let BB' and CC' be the respective internal angle bisectors of angles B and C in triangle ABC, and let a, b, and c denote the sidelengths of the triangle. We set

$$\angle B = 2\beta,\quad \angle C = 2\gamma,\ u = AB',\ U = B'C,\ v = AC',\ V = C'B.$$

We shall see that the assumptions $BB' = CC'$ and $C > B$ (and hence $c > b$) lead to the contradiction that

$$\frac{b}{u} < \frac{c}{v} \ \text{ and } \ \frac{b}{u} \geq \frac{c}{v}.$$

Geometrically, this means that the line $B'C'$ intersects both rays BC and CB. On the one hand, we have

$$\frac{b}{u}-\frac{c}{v}=\frac{u+U}{u}-\frac{v+V}{v}=\frac{U}{u}-\frac{V}{v}=\frac{a}{c}-\frac{a}{b}<0$$

or

$$\frac{b}{u}<\frac{c}{v}.$$

On the other hand, we use the identity $\sin 2\omega = 2\sin\omega\cos\omega$ to obtain

$$\begin{aligned}
\frac{b}{c}\cdot\frac{v}{u} &= \frac{\sin B}{\sin C}\cdot\frac{v}{u}\\
&= \frac{2\cos\beta\sin\beta}{2\cos\gamma\sin\gamma}\cdot\frac{v}{u}\\
&= \frac{\cos\beta}{\cos\gamma}\cdot\frac{\sin\beta}{u}\cdot\frac{v}{\sin\gamma}\\
&= \frac{\cos\beta}{\cos\gamma}\cdot\frac{\sin A}{BB'}\cdot\frac{CC'}{\sin A}\\
&= \frac{\cos\beta}{\cos\gamma}.
\end{aligned}$$

It thus follows that $\frac{b}{u}>\frac{c}{v}$. We meet a contradiction. □

The next inequality is probably the most beautiful 'modern' geometric inequality in triangle geometry.

Theorem 3.9. (The Erdős-Mordell Theorem) If from a point P inside a given triangle ABC perpendiculars PH_1, PH_2, PH_3 are drawn to its sides, then

$$PA+PB+PC\geq 2(PH_1+PH_2+PH_3).$$

This was conjectured by Paul Erdős in 1935, and first proved by Mordell in the same year. Several proofs of this inequality have been given, using Ptolemy's Theorem by André Avez, angular computations with similar triangles by Leon Bankoff, area inequality by V. Komornik, or using trigonometry by Mordell and Barrow.

Proof. [MB] We transform it to a trigonometric inequality. Let $h_1 = PH_1$, $h_2 = PH_2$ and $h_3 = PH_3$.

Apply the Since Law and the Cosine Law to obtain

$$\begin{aligned}
PA\sin A = H_2H_3 &= \sqrt{{h_2}^2+{h_3}^2-2h_2h_3\cos(\pi-A)},\\
PB\sin B = H_3H_1 &= \sqrt{{h_3}^2+{h_1}^2-2h_3h_1\cos(\pi-B)},\\
PC\sin C = H_1H_2 &= \sqrt{{h_1}^2+{h_2}^2-2h_1h_2\cos(\pi-C)}.
\end{aligned}$$

So, we need to prove that

$$\sum_{\text{cyclic}}\frac{1}{\sin A}\sqrt{{h_2}^2+{h_3}^2-2h_2h_3\cos(\pi-A)}\geq 2(h_1+h_2+h_3).$$

The main trouble is that the left hand side has too heavy terms with square root expressions. Our strategy is to find a lower bound without square roots. To this end, we express the terms inside the square root as the sum of two squares.

$$\begin{aligned}
{H_2H_3}^2 &= {h_2}^2+{h_3}^2-2h_2h_3\cos(\pi-A)\\
&= {h_2}^2+{h_3}^2-2h_2h_3\cos(B+C)\\
&= {h_2}^2+{h_3}^2-2h_2h_3(\cos B\cos C-\sin B\sin C).
\end{aligned}$$

Using $\cos^2 B + \sin^2 B = 1$ and $\cos^2 C + \sin^2 C = 1$, one finds that

$$H_2H_3{}^2 = (h_2 \sin C + h_3 \sin B)^2 + (h_2 \cos C - h_3 \cos B)^2 .$$

Since $(h_2 \cos C - h_3 \cos B)^2$ is clearly nonnegative, we get $H_2H_3 \geq h_2 \sin C + h_3 \sin B$. Hence,

$$\begin{aligned}
\sum_{\text{cyclic}} \frac{\sqrt{h_2{}^2 + h_3{}^2 - 2h_2h_3 \cos(\pi - A)}}{\sin A} &\geq \sum_{\text{cyclic}} \frac{h_2 \sin C + h_3 \sin B}{\sin A} \\
&= \sum_{\text{cyclic}} \left(\frac{\sin B}{\sin C} + \frac{\sin C}{\sin B} \right) h_1 \\
&\geq \sum_{\text{cyclic}} 2\sqrt{\frac{\sin B}{\sin C} \cdot \frac{\sin C}{\sin B}} h_1 \\
&= 2h_1 + 2h_2 + 2h_3.
\end{aligned}$$

□

Epsilon 38. [SL 2005 KOR] *In an acute triangle ABC, let D, E, F, P, Q, R be the feet of perpendiculars from A, B, C, A, B, C to BC, CA, AB, EF, FD, DE, respectively. Prove that*

$$p(ABC)p(PQR) \geq p(DEF)^2,$$

where $p(T)$ denotes the perimeter of triangle T .

Epsilon 39. [IMO 2001/1 KOR] *Let ABC be an acute-angled triangle with O as its circumcenter. Let P on line BC be the foot of the altitude from A. Assume that $\angle BCA \geq \angle ABC + 30^\circ$. Prove that $\angle CAB + \angle COP < 90^\circ$.*

Epsilon 40. [IMO 1961/2 POL] (Weitzenböck's Inequality) Let a, b, c be the lengths of a triangle with area S. Show that

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}S.$$

Epsilon 41. (The Neuberg-Pedoe Inequality) Let a_1, b_1, c_1 denote the sides of the triangle $A_1B_1C_1$ with area F_1. Let a_2, b_2, c_2 denote the sides of the triangle $A_2B_2C_2$ with area F_2. Then, we have

$$a_1{}^2(b_2{}^2 + c_2{}^2 - a_2{}^2) + b_1{}^2(c_2{}^2 + a_2{}^2 - b_2{}^2) + c_1{}^2(a_2{}^2 + b_2{}^2 - c_2{}^2) \geq 16F_1F_2.$$

We close this subsection with Barrows' Inequality stronger than The Erdös-Mordell Theorem. We need the following trigonometric inequality:

Proposition 3.6. (Wolstenholme's Inequality) Let $x, y, z, \theta_1, \theta_2, \theta_3$ be real numbers with $\theta_1 + \theta_2 + \theta_3 = \pi$. Then, the following inequality holds:

$$x^2 + y^2 + z^2 \geq 2(yz \cos \theta_1 + zx \cos \theta_2 + xy \cos \theta_3).$$

Proof. Using $\theta_3 = \pi - (\theta_1 + \theta_2)$, we have the identity

$$\begin{aligned}
& x^2 + y^2 + z^2 - 2(yz \cos \theta_1 + zx \cos \theta_2 + xy \cos \theta_3) \\
= \quad & [\ z - (x \cos \theta_2 + y \cos \theta_1)\ ]^2 + [\ x \sin \theta_2 - y \sin \theta_1\ ]^2 .
\end{aligned}$$

□

Corollary 3.1. *Let p, q, and r be positive real numbers. Let θ_1, θ_2, and θ_3 be real numbers satisfying $\theta_1 + \theta_2 + \theta_3 = \pi$. Then, the following inequality holds.*

$$p \cos \theta_1 + q \cos \theta_2 + r \cos \theta_3 \leq \frac{1}{2} \left(\frac{qr}{p} + \frac{rp}{q} + \frac{pq}{r} \right).$$

Proof. Take $(x, y, z) = \left(\sqrt{\frac{qr}{p}}, \sqrt{\frac{rp}{q}}, \sqrt{\frac{pq}{r}} \right)$ and apply the above proposition. □

Delta 26. (Cosmin Pohoaţă) Let a, b, c be the sidelengths of a given triangle ABC with circumradius R, and let x, y, z be three arbitrary real numbers. Then, we have that

$$R\left(\sqrt{\frac{yz}{x}}+\sqrt{\frac{zx}{y}}+\sqrt{\frac{xy}{z}}\right)\geq\sqrt{xa^2+yb^2+zc^2}.$$

Epsilon 42. (Barrow's Inequality) Let P be an interior point of a triangle ABC and let U, V, W be the points where the bisectors of angles BPC, CPA, APB cut the sides BC,CA,AB respectively. Then, we have

$$PA+PB+PC\geq 2(PU+PV+PW).$$

Epsilon 43. [AK] Let $x_1,\cdots,x_4$ be positive real numbers. Let $\theta_1,\cdots,\theta_4$ be real numbers such that $\theta_1+\cdots+\theta_4=\pi$. Then, we have

$$x_1\cos\theta_1+x_2\cos\theta_2+x_3\cos\theta_3+x_4\cos\theta_4\leq\sqrt{\frac{(x_1x_2+x_3x_4)(x_1x_3+x_2x_4)(x_1x_4+x_2x_3)}{x_1x_2x_3x_4}}.$$

Delta 27. [RS] Let $R,r,s>0$. Show that a necessary and sufficient condition for the existence of a triangle with circumradius R, inradius r, and semiperimeter s is

$$s^4-2(2R^2+10Rr-r^2)s^2+r(4R+r)^2\leq 0.$$

3.5. **Erdős, Brocard, and Weitzenböck.** In this section, we touch Brocard geometry. We begin with a consequence of The Erdős-Mordell Theorem.

Epsilon 44. [IMO 1991/5 FRA] *Let ABC be a triangle and P an interior point in ABC. Show that at least one of the angles $\angle PAB$, $\angle PBC$, $\angle PCA$ is less than or equal to $30°$.*

As an immediate consequence, one may consider the following symmetric situation:

Proposition 3.7. *Let ABC be a triangle. If there exists an interior point P in ABC satisfying that*

$$\angle PAB = \angle PBC = \angle PCA = \omega$$

for some positive real number ω. Then, we have the inequality $\omega \leq \frac{\pi}{6}$.

We omit the geometrical proof of the existence and the uniqueness of such point in an arbitrary triangle.(Prove it!)

Delta 28. *Let ABC be a triangle. There exists a unique interior point Ω_1, which bear the name the first Brocard point of ABC, such that*

$$\angle \Omega_1 AB = \angle \Omega_1 BC = \angle \Omega_1 CA = \omega_1$$

for some ω_1, the first Brocard angle.

By symmetry, we also include

Delta 29. *Let ABC be a triangle. There exists a unique interior point Ω_2 with*

$$\angle \Omega_2 BA = \angle \Omega_2 CB = \angle \Omega_2 AC = \omega_2$$

for some ω_2, the second Brocard angle. The point Ω_2 is called the second Brocard point of ABC.

Delta 30. *If a triangle ABC has an interior point P such that $\angle PAB = \angle PBC = \angle PCA = 30°$, then it is equilateral.*

Epsilon 45. *Any triangle has the same Brocard angles.*

As a historical remark, we state that H. Brocard (1845-1922) was not the first one who discovered the Brocard points. They were also known to A. Crelle (1780-1855), C. Jacobi (1804-1851), and others some 60 years earlier. However, their results in this area were soon forgotten [RH]. Our next job is to evaluate the Brocard angle quite explicitly.

Epsilon 46. *The Brocard angle ω of the triangle ABC satisfies*

$$\cot\omega = \cot A + \cot B + \cot C.$$

Proposition 3.8. *The Brocard angle ω of the triangle with sides a, b, c and area S satisfies*

$$\cot\omega = \frac{a^2+b^2+c^2}{4S}.$$

Proof. We have

$$\begin{aligned}
\cot A + \cot B + \cot C &= \frac{2bc\cos A}{2bc\sin A} + \frac{2ca\cos B}{2ca\sin B} + \frac{2ab\cos C}{2ab\sin C} \\
&= \frac{b^2+c^2-a^2}{4S} + \frac{c^2+a^2-b^2}{4S} + \frac{a^2+b^2-c^2}{4S} \\
&= \frac{a^2+b^2+c^2}{4S}.
\end{aligned}$$

□

We revisit Weitzenböck's Inequality. It is a corollary of The Erdős-Mordell Theorem!

Proposition 3.9. [IMO 1961/2 POL] (Weitzenböck's Inequality) Let a, b, c be the lengths of a triangle with area S. Show that

$$a^2+b^2+c^2 \geq 4\sqrt{3}S.$$

Third Proof. Letting ω denote its Brocard angle, by combining results we proved, we obtain

$$\frac{a^2+b^2+c^2}{4S} = \cot\omega \geq \cot\left(\frac{\pi}{6}\right) = \sqrt{3}.$$

□

We present interesting theorems from Brocard geometry.

Delta 31. [RH] Let Ω_1 and Ω_2 denote the Brocard points of a triangle ABC with the circumcenter O. Let the circumcircle of $O\Omega_1\Omega_2$, called the Brocard circle of ABC, meet the line $A\Omega_1$, $B\Omega_1$, $C\Omega_1$ at R, P, Q, respectively, again. The triangle PQR bears the name the first Brocard triangle of ABC.

(a) $O\Omega_1 = O\Omega_2$.

(b) Two triangles PQR and ABC are similar.

(c) Two triangles PQR and ABC have the same centroid.

(d) Let U, V, W denote the midpoints of QR, RP, PQ, respectively. Let U_H, V_H, W_H denote the feet of the perpendiculars from U, V, W respectively. Then, the three lines UU_H, VU_H, WW_H meet at the nine point circle of triangle ABC.

The story is not over. We establish an inequality which implies the problem **[IMO 1991/5 FRA]**.

Epsilon 47. (The Trigonometric Versions of Ceva's Theorem) For an interior point P of a triangle $A_1A_2A_3$, we write

$$\angle A_3A_1A_2 = \alpha_1,\ \angle PA_1A_2 = \vartheta_1,\ \angle PA_1A_3 = \theta_1,$$
$$\angle A_1A_2A_3 = \alpha_2,\ \angle PA_2A_3 = \vartheta_2,\ \angle PA_2A_1 = \theta_2,$$
$$\angle A_2A_3A_1 = \alpha_3,\ \angle PA_3A_1 = \vartheta_3,\ \angle PA_3A_2 = \theta_3.$$

Then, we find a hidden symmetry:

$$\frac{\sin\vartheta_1}{\sin\theta_1}\cdot\frac{\sin\vartheta_2}{\sin\theta_2}\cdot\frac{\sin\vartheta_3}{\sin\theta_3} = 1,$$

or equivalently,

$$\frac{1}{\sin\alpha_1\sin\alpha_2\sin\alpha_3} = \left[\cot\vartheta_1 - \cot\alpha_1\right]\left[\cot\vartheta_2 - \cot\alpha_2\right]\left[\cot\vartheta_3 - \cot\alpha_3\right].$$

Epsilon 48. *Let P be an interior point of a triangle ABC. Show that*

$$\cot(\angle PAB) + \cot(\angle PBC) + \cot(\angle PCA) \geq 3\sqrt{3}.$$

Proposition 3.10. [IMO 1991/5 FRA] *Let ABC be a triangle and P an interior point in ABC. Show that at least one of the angles $\angle PAB$, $\angle PBC$, $\angle PCA$ is less than or equal to 30°.*

Second Solution. The above inequality implies

$$max\{\ \cot(\angle PAB),\ \cot(\angle PBC),\ \cot(\angle PCA)\ \} \geq \sqrt{3} = \cot 30^\circ.$$

Since the cotangent function is strictly decreasing on $(0,\pi)$, we get the result. □

3.6. **From Incenter to Centroid.** We begin with an inequality regarding the incenter. In fact, the geometric inequality is equivalent to an algebraic one, Schur's Inequality!

Example 7. (Korea 1998) Let I be the incenter of a triangle ABC. Prove that

$$IA^2 + IB^2 + IC^2 \geq \frac{BC^2 + CA^2 + AB^2}{3}.$$

Proof. Let $BC = a$, $CA = b$, $AB = c$, and $s = \frac{a+b+c}{2}$. Letting r denote the inradius of $\triangle ABC$, we have

$$r^2 = \frac{(s-a)(s-b)(s-c)}{s}.$$

By The Pythagoras Theorem, the inequality is equivalent to

$$(s-a)^2 + r^2 + (s-b)^2 + r^2 + (s-c)^2 + r^2 \geq \frac{1}{3}\left(a^2 + b^2 + c^2\right).$$

or

$$(s-a)^2 + (s-b)^2 + (s-c)^2 + \frac{3(s-a)(s-b)(s-c)}{s} \geq \frac{1}{3}\left(a^2 + b^2 + c^2\right).$$

After The Ravi Substitution $x = s-a$, $y = s-b$, $z = s-c$, it becomes

$$x^2 + y^2 + z^2 + \frac{3xyz}{x+y+z} \geq \frac{(x+y)^2 + (y+z)^2 + (z+x)^2}{3}$$

or

$$3\left(x^2 + y^2 + z^2\right)(x+y+z) + 9xyz \geq (x+y+z)\left((x+y)^2 + (y+z)^2 + (z+x)^2\right)$$

or

$$9xyz \geq (x+y+z)\left(2xy + 2yz + 2zx - x^2 - y^2 - z^2\right)$$

or

$$9xyz \geq x^2y + x^2z + y^2z + y^2x + z^2x + z^2y + 6xyz - x^3 - y^3 - z^3$$

or

$$x^3 + y^3 + z^3 + 3xyz \geq x^2(y+z) + y^2(z+x) + z^2(x+y).$$

This is a particular case of Schur's Inequality. □

Now, one may ask more questions. Can we replace the incenter by other classical points in triangle geometry? The answer is yes. We first take the centroid.

Example 8. *Let G denote the centroid of the triangle ABC. Then, we have the geometric identity*

$$GA^2 + GB^2 + GC^2 = \frac{BC^2 + CA^2 + AB^2}{3}.$$

Proof. Let M denote the midpoint of BC. The Pappus Theorem implies that

$$\frac{GB^2 + GC^2}{2} = GM^2 + \left(\frac{BC}{2}\right)^2 = \left(\frac{GA}{2}\right)^2 + \left(\frac{BC}{2}\right)^2$$

or

$$-GA^2 + 2GB^2 + 2GC^2 = BC^2$$

Similarly, $2GA^2 - GB^2 + 2GC^2 = CA^2$ and $2GA^2 + 2GB^2 - 2GC^2 = AB^2$. Adding these three equalities, we get the identity. □

Before we take other classical points, we need to rethink this unexpected situation. We have an equality, instead of an inequality. According to this equality, we find that the previous inequality can be rewritten as

$$IA^2 + IB^2 + IC^2 \geq GA^2 + GB^2 + GC^2.$$

Now, it is quite reasonable to make a conjecture which states that, given a triangle ABC, the minimum value of $PA^2 + PB^2 + PC^2$ is attained when P is the centroid of $\triangle ABC$. This guess is true!

Theorem 3.10. *Let $A_1A_2A_3$ be a triangle with the centroid G. For any point P, we have*

$$PA_1{}^2 + PA_2{}^2 + PA_3{}^2 \geq GA^2 + GB^2 + GC^2.$$

Proof. Just toss the picture on the real plane $\mathbb{R}^2$ so that

$$P(p,q),\quad A_1(x_1,y_1),\quad A_2(x_2,y_2),\quad A_3(x_3,y_3),\quad G\left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3}\right).$$

What we need to do is to compute

$$\begin{aligned}
&3\left(PA_1{}^2 + PA_2{}^2 + PA_3{}^2\right) - \left(BC^2 + CA^2 + AB^2\right)\\
=\;&3\sum_{i=1}^{3}(p-x_i)^2 + (q-y_i)^2 - \sum_{i=1}^{3}\left(\frac{x_1+x_2+x_3}{3} - x_i\right)^2 + \left(\frac{y_1+y_2+y_3}{3} - y_i\right)^2\\
=\;&3\sum_{i=1}^{3}(p-x_i)^2 - \sum_{i=1}^{3}\left(\frac{x_1+x_2+x_3}{3} - x_i\right)^2 + 3\sum_{i=1}^{3}(q-y_i)^2 - \sum_{i=1}^{3}\left(\frac{y_1+y_2+y_3}{3} - y_i\right)^2.
\end{aligned}$$

A moment's thought shows that the quadratic polynomials are squares.

$$3\sum_{i=1}^{3}(p-x_i)^2 - \sum_{i=1}^{3}\left(\frac{x_1+x_2+x_3}{3} - x_i\right)^2 = 9\left(p - \frac{x_1+x_2+x_3}{3}\right)^2,$$

$$3\sum_{i=1}^{3}(q-y_i)^2 - \sum_{i=1}^{3}\left(\frac{y_1+y_2+y_3}{3} - y_i\right)^2 = 9\left(q - \frac{y_1+y_2+y_3}{3}\right)^2.$$

Hence, the quantity $3\left(PA_1{}^2 + PA_2{}^2 + PA_3{}^2\right) - \left(BC^2 + CA^2 + AB^2\right)$ is clearly non-negative. Furthermore, we notice that the above proof of the geometric inequality **discovers** a geometric identity:

$$\left(PA^2 + PB^2 + PC^2\right) - \left(GA^2 + GB^2 + GC^2\right) = 9GP^2.$$

It is clear that the equality in the above inequality holds only when $GP = 0$ or $P = G$. □

After removing the special condition that P is the incenter, we get a more general inequality, even without using a heavy machine, like Schur's Inequality. Sometimes, generalizations are more easy! Taking the point P as the circumcenter, we have

Proposition 3.11. *Let ABC be a triangle with circumradius R. Then, we have*

$$AB^2 + BC^2 + CA^2 \leq 9R^2.$$

Proof. Let O and G denote its circumcenter and centroid, respectively. It reads

$$9GO^2 + \left(AB^2 + BC^2 + CA^2\right) = 3\left(OA^2 + OB^2 + OC^2\right) = 9R^2.$$

□

The readers can rediscover many geometric inequalities by taking other classical points from triangle geometry.(Do it!) Here goes another inequality regarding the incenter.

Example 9. *Let I be the incenter of the triangle ABC with $BC = a$, $CA = b$ and $AB = c$. Prove that, for all points X,*

$$aXA^2 + bXB^2 + cXC^2 \geq abc.$$

First Solution. It turns out that the non-negative quantity

$$aXA^2 + bXB^2 + cXC^2 - abc$$

has a geometric meaning. This geometric inequality follows from the following geometric identity:

$$aXA^2 + bXB^2 + cXC^2 = (a+b+c)XI^2 + abc.$$ [7]

[7] **[SL 1988 SGP]**

There are many ways to establish this identity. To euler[8] it, we toss the picture on the real plane $\mathbb{R}^2$ with the coordinates

$$A(c\cos B, c\sin B),\quad B(0,0),\quad C(a,0).$$

Let r denote the inradius of $\triangle ABC$. Setting $s=\frac{a+b+c}{2}$, we get $I(s-b,r)$. It is well-known that

$$r^2=\frac{(s-a)(s-b)(s-c)}{s}.$$

Set $X(p,q)$. On the one hand, we obtain

$$\begin{aligned}
& aXA^2+bXB^2+cXC^2\\
= \;& a\left[(p-c\cos B)^2+(q-c\sin B)^2\right]+b\left(p^2+q^2\right)+c\left[(p-a)^2+q^2\right]\\
= \;& (a+b+c)p^2-2acp(1+\cos B)+(a+b+c)q^2-2acq\sin B+ac^2+a^2c\\
= \;& 2sp^2-2acp\left(1+\frac{a^2+c^2-b^2}{2ac}\right)+2sq^2-2acq\frac{[\triangle ABC]}{\frac{1}{2}ac}+ac^2+a^2c\\
= \;& 2sp^2-p(a+c+b)\left(a+c-b\right)+2sq^2-4q[\triangle ABC]+ac^2+a^2c\\
= \;& 2sp^2-p(2s)\left(2s-2b\right)+2sq^2-4qsr+ac^2+a^2c\\
= \;& 2sp^2-4s\left(s-b\right)p+2sq^2-4rsq+ac^2+a^2c.
\end{aligned}$$

On the other hand, we obtain

$$\begin{aligned}
& (a+b+c)XI^2+abc\\
= \;& 2s\left[(p-(s-b))^2+(q-r)^2\right]\\
= \;& 2s\left[p^2-2(s-b)p+(s-b)^2+q^2-2qr+r^2\right]\\
= \;& 2sp^2-4s\left(s-b\right)p+2s(s-b)^2+2sq^2-4rsq+2sr^2+abc.
\end{aligned}$$

It thus follows that

$$\begin{aligned}
& aXA^2+bXB^2+cXC^2-(a+b+c)XI^2-abc\\
= \;& ac^2+a^2c-2s(s-b)^2-2sr^2-abc\\
= \;& ac(a+c)-2s(s-b)^2-2(s-a)(s-b)(s-c)-abc\\
= \;& ac(a+c-b)-2s(s-b)^2-2(s-a)(s-b)(s-c)\\
= \;& 2ac(s-b)-2s(s-b)^2-2(s-a)(s-b)(s-c)\\
= \;& 2(s-b)\left[ac-s(s-b)-2(s-a)(s-c)\right].
\end{aligned}$$

However, we compute $ac-s(s-b)-2(s-a)(s-c)=-2s^2+(a+b+c)s=0$. □

Now, throw out the special condition that I is the incenter! Then, the essence appears:

Delta 32. (The Leibniz Theorem) Let $\omega_1,\omega_2,\omega_3$ be real numbers such that $\omega_1+\omega_2+\omega_3\neq 0$. We characterize the generalized centroid $G_\omega=G_{(\omega_1,\omega_2,\omega_3)}$ by

$$\overrightarrow{XG_\omega}=\sum_{i=1}^{3}\frac{\omega_i}{\omega_1+\omega_2+\omega_3}\overrightarrow{XA_i}.$$

Then G_ω is well-defined in the sense that it doesn't depend on the choice of X. For all points P, we have

$$\sum_{i=1}^{3}\omega_i {PA_i}^2=(\omega_1+\omega_2+\omega_3){PG_\omega}^2+\sum_{i=1}^{3}\frac{\omega_i\omega_{i+1}}{\omega_1+\omega_2+\omega_3}{A_iA_{i+1}}^2.$$

We show that the geometric identity $aXA^2+bXB^2+cXC^2=(a+b+c)XI^2+abc$ is a straightforward consequence of The Leibniz Theorem.

[8]euler v. (in Mathematics) transform the geometric identity in triangle geometry to trigonometric or algebraic identity.

Second Solution. Let $BC = a, CA = b, AB = c$. With the weights (a, b, c), we have $I = G_{(a,b,c)}$. Hence,

$$\begin{aligned} aXA^2 + bXB^2 + cXC^2 &= (a+b+c)XI^2 + \frac{bc}{a+b+c}a^2 + + \frac{ca}{a+b+c}b^2 + \frac{ab}{a+b+c}c^2 \\ &= (a+b+c)XI^2 + abc. \end{aligned}$$

□

Epsilon 49. [IMO 1961/2 POL] (Weitzenböck's Inequality) Let a, b, c be the lengths of a triangle with area S. Show that

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}S.$$

Epsilon 50. (The Neuberg-Pedoe Inequality) Let a_1, b_1, c_1 denote the sides of the triangle $A_1B_1C_1$ with area F_1. Let a_2, b_2, c_2 denote the sides of the triangle $A_2B_2C_2$ with area F_2. Then, we have

$${a_1}^2({b_2}^2 + {c_2}^2 - {a_2}^2) + {b_1}^2({c_2}^2 + {a_2}^2 - {b_2}^2) + {c_1}^2({a_2}^2 + {b_2}^2 - {c_2}^2) \geq 16F_1F_2.$$

Delta 33. [SL 1988 UNK] *The triangle ABC is acute-angled. Let L be any line in the plane of the triangle and let u, v, w be lengths of the perpendiculars from A, B, C respectively to L. Prove that*

$$u^2 \tan A + v^2 \tan B + w^2 \tan C \geq 2\triangle,$$

where $\triangle$ is the area of the triangle, and determine the lines L for which equality holds.

Delta 34. [KWL] Let G and I be the centroid and incenter of the triangle ABC with inradius r, semiperimeter s, circumradius R. Show that

$$IG^2 = \frac{1}{9}\left(s^2 + 5r^2 - 16Rr\right).$$

Inspiration is needed in geometry, just as much as in poetry. - A. Pushkin

4. Geometry Revisited

It gives me the same pleasure when someone else proves a good theorem as when I do it myself.

- E. Landau

4.1. **Areal Co-ordinates.**

In this section we aim to briefly introduce develop the theory of areal (or 'barycentric') co-ordinate methods with a view to making them accessible to a reader as a means for solving problems in plane geometry. Areal co-ordinate methods are particularly useful and important for solving problems based upon a triangle, because, unlike Cartesian co-ordinates, they exploit the natural symmetries of the triangle and many of its key points in a very beautiful and useful way.

4.1.1. *Setting up the co-ordinate system.*

If we are going to solve a problem using areal co-ordinates, the first thing we must do is choose a triangle ABC, which we call the *triangle of reference*, and which plays a similar role to the axes in a cartesian co-ordinate system. Once this triangle is chosen, we can assign to each point P in the plane a unique triple (x, y, z) fixed such that $x + y + z = 1$, which we call the areal co-ordinates of P. The way these numbers are assigned can be thought of in three different ways, all of which are useful in different circumstances. We leave the proofs that these three conditions are equivalent, along with a proof of the uniqueness of areal co-ordinate representation, for the reader. The first definition we shall see is probably the most intuitive and most useful for working with. It also explains why they are known as 'areal' co-ordinates.

1st Definition: A point P internal to the triangle ABC has areal co-ordinates

$$\left(\frac{[PBC]}{[ABC]}, \frac{[PCA]}{[ABC]}, \frac{[PAB]}{[ABC]}\right).$$

If a sign convention is adopted, such that a triangle whose vertices are labelled clockwise has negative area, this definition applies for all P in the plane.

2nd Definition: If x, y, z are the masses we must place at the vertices A, B, C respectively such that the resulting system has centre of mass P, then (x, y, z) are the areal co-ordinates of P (hence the alternative name 'barycentric')

3rd Definition: If we take a system of vectors with arbitrary origin (not on the sides of triangle ABC) and let **a, b, c, p** be the position vectors of A, B, C, P respectively, then $\mathbf{p} = x\mathbf{a} + y\mathbf{b} + z\mathbf{c}$ for some triple (x, y, z) such that $x + y + z = 1$. We define this triple as the areal co-ordinates of P.

There are some remarks immediately worth making:

- The vertices A, B, C of the triangle of reference have co-ordinates $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ respectively.
- All the co-ordinates of a point are positive if and only if the point lies within the triangle of reference, and if any of the co-ordinates are zero, the point lies on one of the sides (or extensions of the sides) of ABC.

4.1.2. *The Equation of a Line.* A line is a geometrical object such that any pair of non-parallel lines meet at one and only one point. We would therefore expect the equation of a line to be linear, such that any pair of simultaneous line equations, together with the condition $x + y + z = 1$, can be solved for a unique triple (x, y, z) corresponding to the areal co-ordinates of the point of intersection of the two lines. Indeed, it follows (using the equation $x + y + z = 1$ to eliminate any constant terms) that the general equation of a line is of the form

$$lx + my + nz = 0$$

where l, m, n are constants and not all zero. Clearly there exists a *unique* line (up to multiplication by a constant) containing any two given points $P(x_p, y_p, z_p), Q(x_q, y_q, z_q)$. This line can be written explicitly as

$$(y_p z_q - y_q z_p)x + (z_p x_q - z_q x_p)y + (x_p y_q - x_q y_p)z = 0$$

This equation is perhaps more neatly expressed in the determinant[9] form:

$$\mathrm{Det}\begin{pmatrix} x & x_p & x_q \\ y & y_p & y_q \\ z & z_p & z_q \end{pmatrix} = 0.$$

While the above form is useful, it is often quicker to just spot the line automatically. For example try to spot the equation of the line BC, containing the points $B(0, 1, 0)$ and $C(0, 0, 1)$, without using the above equation.

Of particular interest (and simplicity) are *Cevian* lines, which pass through the vertices of the triangle of reference. We define a **Cevian through A** as a line whose equation is of the form $my = nz$. Clearly any line containing A must have this form, because setting $y = z = 0, x = 1$ any equation with a nonzero x coefficient would not vanish. It is easy to see that any point on this line therefore has form $(x, y, z) = (1 - mt - nt, nt, mt)$ where t is a parameter. In particular, it will intersect the side BC with equation $x = 0$ at the point $U(0, \frac{n}{m+n}, \frac{m}{m+n})$. Note that from definition 1 (or 3) of areal co-ordinates, this implies that the ratio $\frac{BU}{UC} = \frac{[ABU]}{[AUC]} = \frac{m}{n}$.

4.1.3. *Example: Ceva's Theorem.* We are now in a position to start using areal co-ordinates to prove useful theorems. In this section we shall state and prove (one direction of) an important result of Euclidean geometry known as Ceva's Theorem. The author recommends a keen reader only reads the statement of Ceva's theorem initially and tries to prove it for themselves using the ideas introduced above, before reading the proof given.

[9]The Determinant of a 3×3 Matrix. Matrix determinants play an important role in areal co-ordinate methods. We define the **determinant** of a 3 by 3 square matrix A as

$$\mathrm{Det}(A) = \mathrm{Det}\begin{pmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{pmatrix} = a_x(b_y c_z - b_z c_y) + a_y(b_z c_x - bx_c z) + a_z(b_x c_y - b_y c_x).$$

This can be thought of as (as the above equation suggests) multiplying each element of the first column by the determinants of 2x2 matrices formed in the 2nd and 3rd columns and the rows not containing the element of the first column. Alternatively, if you think of the matrix as wrapping around (so b_x is in some sense directly beneath b_z in the above matrix) you can simply take the sum of the products of diagonals running from top-left to bottom-right and subtract from it the sum of the products of diagonals running from bottom-left to top-right (so think of the above RHS as $(a_x b_y c_z + a_y b_z c_x + a_z b_x c_y) - (a_z b_y c_x + a_x b_z c_y + a_y b_x c_z)$). In any case, it is worth making sure you are able to quickly evaluate these determinants if you are to be successful with areal co-ordinates.

Theorem 4.1. (Ceva's Theorem) Let ABC be a triangle and let L, M, N be points on the sides BC, CA, AB respectively. Then the cevians AL, BM, CN are concurrent at a point P if and only if

$$\frac{BL}{LC} \cdot \frac{CM}{MA} \cdot \frac{AN}{NB} = 1$$

Proof. Suppose first that the cevians are concurrent at a point P, and let P have areal co-ordinates (p, q, r). Then AL has equation $qz = ry$ (following the discussion of Cevian lines above), so $L\left(0, \frac{q}{q+r}, \frac{r}{q+r}\right)$, which implies $\frac{BL}{LC} = \frac{r}{q}$. Similarly, $\frac{CM}{MA} = \frac{p}{r}, \frac{AN}{NB} = \frac{q}{p}$. Taking their product we get $\frac{BL}{LC} \cdot \frac{CM}{MA} \cdot \frac{AN}{NB} = 1$, proving one direction of the theorem. We leave the converse to the reader. □

The above proof was very typical of many areal co-ordinate proofs. We only had to go through the details for one of the three cevians, and then could say 'similarly' and obtain ratios for the other two by symmetry. This is one of the great advantages of the areal co-ordinate system in solving problems where such symmetries do exist (particularly problems symmetric in a triangle ABC: such that relabelling the triangle vertices would result in the same problem).

4.1.4. *Areas and Parallel Lines.* One might expect there to be an elegant formula for the area of a triangle in areal co-ordinates, given they are a system constructed on areas. Indeed, there is. If PQR is an arbitrary triangle with $P\,(x_p, y_p, z_p)\,, Q\,(x_q, y_q, z_q)\,, R\,(x_r, y_r, z_r)$ then

$$\frac{[PQR]}{[ABC]} = \operatorname{Det}\begin{pmatrix} x_p & x_q & x_r \\ y_p & y_q & y_r \\ z_p & z_q & z_r \end{pmatrix}$$

An astute reader might notice that this seems like a plausible formula, because if P, Q, R are collinear, it tells us that the triangle PQR has area zero, by the line formula already mentioned. It should be noted that the area comes out as negative if the vertices PQR are labelled in the opposite direction to ABC.

It is now fairly obvious what the general equation for a line parallel to a given line passing through two points $(x_1, y_1, z_1), (x_2, y_2, z_2)$ should be, because the area of the triangle formed by any point on such a line and these two points must be constant, having a constant base and constant height. Therefore this line has equation

$$\operatorname{Det}\begin{pmatrix} x & x_1 & x_2 \\ y & y_1 & y_2 \\ z & z_1 & z_2 \end{pmatrix} = k = k(x+y+z),$$

where $k \in \mathbb{R}$ is a constant.

Delta 35. (United Kingdom 2007) Given a triangle ABC and an arbitrary point P internal to it, let the line through P parallel to BC meet AC at M, and similarly let the lines through P parallel to CA,AB meet AB,BC at N,L respectively. Show that

$$\frac{BL}{LC} \cdot \frac{CM}{MA} \cdot \frac{AN}{NB} \leq \frac{1}{8}$$

Delta 36. (Nikolaos Dergiades) Let DEF be the medial triangle of ABC, and P a point with cevian triangle XYZ (with respect to ABC). Find P such that the lines DX, EY, FZ are parallel to the internal bisectors of angles A, B, C, respectively.

4.1.5. *To infinity and beyond.* Before we start looking at some more definite specific useful tools (like the positions of various interesting points in the triangle), we round off the general theory with a device that, with practice, greatly simplifies areal manipulations. Until now we have been acting subject to the constraint that $x+y+z=1$. In reality, if we are just intersecting lines with lines or lines with conics, and not trying to calculate any ratios, it is legitimate to ignore this constraint and to just consider the points (x, y, z) and (kx, ky, kz) as being the same point for all $k \neq 0$. This is because areal co-ordinates are a special case of a more general class of co-ordinates called **projective homogeneous co-ordinates**[10], where here the projective line at infinity is taken to be the line $x+y+z=0$. This system only works if one makes all equations homogeneous (of the same degree in x, y, z), so, for example, $x+y=1$ and $x^2+y=z$ are not homogeneous, whereas $x+y-z=0$ and $a^2yz+b^2zx+c^2xy=0$ are homogeneous. We can therefore, once all our line and conic equations are happily in this form, no longer insist on $x+y+z=1$, meaning points like the incentre $(\frac{a}{a+b+c}, \frac{b}{a+b+c}, \frac{c}{a+b+c})$ can just be written (a, b, c). Such represenataions are called *unnormalised areal co-ordinates* and usually provide a significant advantage for the practical purposes of doing manipulations. However, if any ratios or areas are to be calculated, it is imperative that the co-ordinates are *normalised* again to make $x+y+z=1$. This process is easy: just apply the map

$$(x, y, z) \mapsto \left(\frac{x}{x+y+z}, \frac{y}{x+y+z}, \frac{z}{x+y+z}\right)$$

4.1.6. *Significant areal points and formulae in the triangle.* We have seen that the vertices are given by $A(1,0,0)$, $B(0,1,0)$, $C(0,0,1)$, and the sides by $x=0$,$y=0$,$z=0$. In the section on the equation of a line we examined the equation of a cevian, and this theory can, together with other knowledge of the triangle, be used to give areal expressions for familiar points in Euclidean triangle geometry. We invite the reader to prove some of the facts below as exercises.

- Triangle centroid: $G(1,1,1)$.[11]
- Centre of the inscribed circle: $I(a, b, c)$.[12]
- Centres of escribed circles: $I_a(-a, b, c)$, $I_b(a, -b, c)$, $I_c(a, b, -c)$.
- Symmedian point: $K(a^2, b^2, c^2)$.
- Circumcentre: $O(\sin 2A, \sin 2B, \sin 2C)$.
- Orthocentre: $H(\tan A, \tan B, \tan C)$.
- The isogonal conjugate of $P(x, y, z)$: $P^*\left(\frac{a^2}{x}, \frac{b^2}{y}, \frac{c^2}{z}\right)$.
- The isotomic conjugate of $P(x, y, z)$: $P^t\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right)$.

It should be noted that the rather nasty trigonometric forms of O and H mean that they should be approached using areals with caution, preferably only if the calculations will be relatively simple.

Delta 37. *Let D, E be the feet of the altitudes from A and B respectively, and P, Q the meets of the angle bisectors AI,BI with BC,CA respectively. Show that D,I,E are collinear if and only if P,O,Q are.*

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[10] The author regrets that, in the interests of concision, he is unable to deal with these co-ordinates in this document, but strongly recommends Christopher Bradley's *The Algebra of Geometry*, published by Highperception, as a good modern reference also with a more detailed account of areals and a plethora of applications of the methods touched on in this document. Even better, though only for projectives and lacking in the wealth of fascinating modern examples, is E.A.Maxwell's *The methods of plane projective geometry based on the use of general homogeneous coordinates*, recommended to the present author by the author of the first book.

[11] The midpoints of the sides BC, CA, AB are given by $(0,1,1)$,$(1,0,1)$ and $(1,1,0)$ respectively.

[12] Hint: use the angle bisector theorem.

4.1.7. *Distances and circles.* We finally quickly outline some slightly more advanced theory, which is occasionally quite useful in some problems, We show how to manipulate conics (with an emphasis on circles) in areal co-ordinates, and how to find the distance between two points in areal co-ordinates. These are placed in the same section because the formulae look quite similar and the underlying theory is quite closely related. Derivations can be found in [Bra1].

Firstly, the general equation of a conic in areal co-ordinates is, since a conic is a general equation of the second degree, and areals are a homogeneous system, given by

$$px^2 + qy^2 + rz^2 + 2dyz + 2ezx + 2fxy = 0$$

Since multiplication by a nonzero constant gives the same equation, we have five independent degrees of freedom, and so may choose the coefficients uniquely (up to multiplication by a constant) in such a way as to ensure five given points lie on such a conic.

In Euclidean geometry, the conic we most often have to work with is the circle. The most important circle in areal co-ordinates is the circumcircle of the reference triangle, which has the equation (with a, b, c equal to BC, CA, AB respectively)

$$a^2yz + b^2zx + c^2xy = 0$$

In fact, sharing two infinite points[13] with the above, a general circle is just a variation on this theme, being of the form

$$a^2yz + b^2zx + c^2xy + (x + y + z)(ux + vy + wz) = 0$$

We can, given three points, solve the above equation for u, v, w substituting in the three desired points to obtain the equation for the unique circle passing through them.

Now, the areal distance formula looks very similar to the circumcircle equation. If we have a pair of points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$, which must be normalised, we may define the displacement $PQ : (x_2 - x_1, y_2 - y_1, z_2 - z_1) = (u, v, w)$, and it is this we shall measure the distance of. So the distance of a displacement $PQ(u, v, w), u + v + w = 0$ is given by

$$PQ^2 = -a^2vw - b^2wu - c^2uv$$

Since $u+v+w = 0$ this is, despite the negative signs, always positive unless $u = v = w = 0$.

Delta 38. *Use the vector definition of areal co-ordinates to prove the areal distance formula and the circumcircle formula.*

4.1.8. *Miscellaneous Exercises.* Here we attach a selection of problems compiled by Tim Hennock, largely from UK IMO activities in 2007 and 2008. None of them are trivial, and some are quite difficult. Good luck!

Delta 39. (UK Pre-IMO training 2007) Let ABC be a triangle. Let D, E, F be the reflections of A, B, C in BC, AC, AB respectively. Show that D, E, F are collinear if and only if $OH = 2R$.

Delta 40. (Balkan MO 2005) Let ABC be an acute-angled triangle whose inscribed circle touches AB and AC at D and E respectively. Let X and Y be the points of intersection of the bisectors of the angles $\angle ACB$ and $\angle ABC$ with the line DE and let Z be the midpoint of BC. Prove that the triangle XYZ is equilateral if and only if $\angle A = 60°$

[13]All circles have two (imaginary) points in common on the line at infinity. It follows that if a conic is a circle, its behaviour at the line at infinity $x + y + z = 0$ must be the same as that of the circumcircle, hence the equation given.

Delta 41. (United Kingdom 2007) Triangle ABC has circumcentre O and centroid M. The lines OM and AM are perpendicular. Let AM meet the circumcircle of ABC again at A'. Lines CA' and AB intersect at D and BA' and AC intersect at E. Prove that the circumcentre of triangle ADE lies on the circumcircle of ABC.

Delta 42. [IMO 2007/4] *In triangle ABC the bisector of $\angle BCA$ intersects the circumcircle again at R, the perpendicular bisector of BC at P, and the perpendicular bisector of AC at Q. The midpoint of BC is K and the midpoint of AC is L. Prove that the triangles RPK and RQL have the same area.*

Delta 43. (RMM 2008) Let ABC be an equilateral triangle. P is a variable point internal to the triangle, and its perpendicular distances to the sides are denoted by a^2, b^2 and c^2 for positive real numbers a, b and c. Find the locus of points P such that a, b and c can be the side lengths of a non-degenerate triangle.

Delta 44. [SL 2006] *Let ABC be a triangle such that $\angle C < \angle A < \frac{\pi}{2}$. Let D be on AC such that $BD = BA$. The incircle of ABC touches AB at K and AC at L. Let J be the incentre of triangle BCD. Prove that KL bisects AJ.*

Delta 45. (United Kingdom 2007) The excircle of a triangle ABC touches the side AB and the extensions of the sides BC and CA at points M, N and P, respectively, and the other excircle touches the side AC and the extensions of the sides AB and BC at points S, Q and R, respectively. If X is the intersection point of the lines PN and RQ, and Y the intersection point of RS and MN, prove that the points X, A and Y are collinear.

Delta 46. (Sharygin GMO 2008) Let ABC be a triangle and let the excircle opposite A be tangent to the side BC at A_1. N is the Nagel point of ABC, and P is the point on AA_1 such that $AP = NA_1$. Prove that P lies on the incircle of ABC.

Delta 47. (United Kingdom 2007) Let ABC be a triangle with $\angle B \neq \angle C$. The incircle I of ABC touches the sides BC, CA, AB at the points D, E, F, respectively. Let AD intersect I at D and P. Let Q be the intersection of the lines EF and the line passing through P and perpendicular to AD, and let X, Y be intersections of the line AQ and DE, DF, respectively. Show that the point A is the midpoint of XY.

Delta 48. (Sharygin GMO 2008) Given a triangle ABC. Point A_1 is chosen on the ray BA so that the segments BA_1 and BC are equal. Point A_2 is chosen on the ray CA so that the segments CA_2 and BC are equal. Points B_1, B_2 and C_1, C_2 are chosen similarly. Prove that the lines A_1A_2, B_1B_2 and C_1C_2 are parallel.

4.2. **Concurrencies around Ceva's Theorem.** In this section, we shall present some corollaries and applications of Ceva's theorem.

Theorem 4.2. *Let $\triangle ABC$ be a given triangle and let A_1, B_1, C_1 be three points on lying on its sides BC, CA and AB, respectively. Then, the three lines AA_1 ,BB_1, CC_1 concur if and only if*

$$\frac{A'B}{A'C} \cdot \frac{B'C}{B'A} \cdot \frac{C'A}{C'B} = 1.$$

Proof. We shall resume to proving only the direct implication. After reading the following proof, you will understand why. Denote by P the intersection of the lines AA_1, BB_1, CC_1. The parallel to BC through P meets CA at B_a and AB at C_a. The parallel to CA through P meets AB at C_b and BC at A_b. The parallel to AB through P meets BC at A_c and CA at B_c. As segments on parallels, we get $\frac{C_1A}{C_1B} = \frac{PB_c}{PA_c}$. On the other hand, we get

$$\frac{B_cP}{AB} = \frac{PB_1}{BB_1} \text{ and } \frac{PA_c}{AB} = \frac{PA_1}{AA_1}.$$

It follows that

$$\frac{B_cP}{AB} : \frac{PA_c}{AB} = \frac{PB_1}{BB_1} : \frac{PA_1}{AA_1}$$

so that

$$\frac{B_cP}{PA_c} = \frac{PB_1}{BB_1} : \frac{PA_1}{AA_1}.$$

Consequently, we obtain

$$\frac{C_1A}{C_1B} = \frac{PB_1}{BB_1} : \frac{PA_1}{AA_1}.$$

Similarly, we deduce that $\frac{A_1B}{A_1C} = \frac{PC_1}{CC_1} : \frac{PB_1}{BB_1}$ and $\frac{B_1C}{B_1A} = \frac{PA_1}{AA_1} : \frac{PC_1}{CC_1}$. Now

$$\frac{A'B}{A'C} \cdot \frac{B'C}{B'A} \cdot \frac{C'A}{C'B} = \left(\frac{PC_1}{CC_1} : \frac{PB_1}{BB_1}\right) \cdot \left(\frac{PA_1}{AA_1} : \frac{PC_1}{CC_1}\right) \cdot \left(\frac{PB_1}{BB_1} : \frac{PA_1}{AA_1}\right) = 1,$$

which proves Ceva's theorem. □

Corollary 4.1. (The Trigonometric Version of Ceva's Theorem) In the configuration described above, the lines AA_1, BB_1, CC_1 are concurrent if and only if

$$\frac{\sin A_1AB}{\sin A_1AC} \cdot \frac{\sin C_1CA}{\sin C_1CB} \cdot \frac{\sin B_1BC}{\sin B_1BA} = 1.$$

Proof. By the Sine Law, applied in the triangles A_1AB and A_1AC, we have

$$\frac{A_1B}{\sin A_1AB} = \frac{AB}{\sin AA_1B}, \quad \text{and} \quad \frac{A_1C}{\sin A_1AC} = \frac{AC}{\sin AA_1C}.$$

Hence,

$$\frac{A_1B}{A_1C} = \frac{AB}{AC} \cdot \frac{\sin A_1AB}{\sin A_1AC}.$$

Similarly, $\frac{B_1C}{B_1A} = \frac{BC}{AB} \cdot \frac{\sin B_1BC}{\sin B_1BA}$ and $\frac{C_1A}{C_1B} = \frac{AC}{BC} \cdot \frac{\sin C_1CA}{\sin C_1CB}$. Thus, we conclude that

$$\begin{aligned} & \frac{\sin A_1AB}{\sin A_1AC} \cdot \frac{\sin C_1CA}{\sin C_1CB} \cdot \frac{\sin B_1BC}{\sin B_1BA} \\ = & \left(\frac{A_1B}{A_1C} \cdot \frac{AC}{AB}\right) \cdot \left(\frac{C_1A}{C_1B} \cdot \frac{BC}{AC}\right) \cdot \left(\frac{B_1C}{B_1A} \cdot \frac{AB}{BC}\right) \\ = & 1. \end{aligned}$$

□

We begin now with a result, which most of you might know it as *Jacobi's theorem.*

Proposition 4.1. (Jacobi's Theorem) Let ABC be a triangle, and let X, Y, Z be three points in its plane such that $\angle YAC = \angle BAZ$, $\angle ZBA = \angle CBX$ and $\angle XCB = \angle ACY$. Then, the lines AX, BY, CZ are concurrent.

Proof. We use directed angles taken modulo 180°. Denote by A, B, C, x, y, z the magnitudes of the angles $\angle CAB$, $\angle ABC$, $\angle BCA$, $\angle YAC$, $\angle ZBA$, and $\angle XCB$, respectively. Since the lines AX, BX, CX are (obviously) concurrent (at X), the trigonometric version of Ceva's theorem yields

$$\frac{\sin CAX}{\sin XAB} \cdot \frac{\sin ABX}{\sin XBC} \cdot \frac{\sin BCX}{\sin XCA} = 1.$$

We now notice that

$$\angle ABX = \angle ABC + \angle CBX = B + y, \quad \angle XBC = -\angle CBX = -y,$$

$$\angle BCX = -\angle XCB = -z, \quad \angle XCA = \angle XCB + \angle BCA = z + C.$$

Hence, we get

$$\frac{\sin CAX}{\sin XAB} \cdot \frac{\sin (B+y)}{\sin (-y)} \cdot \frac{\sin (-z)}{\sin (C+z)} = 1.$$

Similarly, we can find

$$\frac{\sin ABY}{\sin YBC} \cdot \frac{\sin (C+z)}{\sin (-z)} \cdot \frac{\sin (-x)}{\sin (A+x)} = 1,$$

$$\frac{\sin BCZ}{\sin ZCA}\cdot\frac{\sin(A+x)}{\sin(-x)}\cdot\frac{\sin(-y)}{\sin(B+y)}=1.$$

Multiplying all these three equations and canceling the same terms, we get

$$\frac{\sin CAX}{\sin XAB}\cdot\frac{\sin ABY}{\sin YBC}\cdot\frac{\sin BCZ}{\sin ZCA}=1.$$

According to the trigonometric version of Ceva's theorem, the lines AX, BY, CZ are concurrent. □

We will see that Jacobi's theorem has many interesting applications. We start with the well-known Karyia theorem.

Theorem 4.3. (Kariya's Theorem) Let I be the incenter of a given triangle ABC, and let D, E, F be the points where the incircle of ABC touches the sides BC, CA, AB. Now, let X, Y, Z be three points on the lines ID, IE, IF such that the directed segments IX, IY, IZ are congruent. Then, the lines AX, BY, CZ are concurrent.

Proof. (Darij Grinberg) Being the points of tangency of the incircle of triangle ABC with the sides AB and BC, the points F and D are symmetric to each other with respect to the angle bisector of the angle ABC, i. e. with respect to the line BI. Thus, the triangles BFI and BDI are inversely congruent. Now, the points Z and X are corresponding points in these two inversely congruent triangles, since they lie on the (corresponding) sides IF and ID of these two triangles and satisfy $IZ = IX$. Corresponding points in inversely congruent triangles form oppositely equal angles, i.e. $\angle ZBF = -\angle XBD$. In other words, $\angle ZBA = \angle CBX$. Similarly, we have that $\angle XCB = \angle ACY$ and $\angle YAC = \angle BAZ$. Note that the points X, Y, Z satisfy the condition from Jacobi's theorem, and therefore, we conclude that the lines AX, BY, CZ are concurrent. □

Another such corollary is the Kiepert theorem, which generalizes the existence of the Fermat points.

Delta 49. (Kiepert's Theorem) Let ABC be a triangle, and let BXC, CYA, AZB be three directly similar isosceles triangles erected on its sides BC, CA, and AB, respectively. Then, the lines AX, BY, CZ concur at one point.

Delta 50. (Floor van Lamoen) Let A', B', C' be three points in the plane of a triangle ABC such that $\angle B'AC = \angle BAC'$, $\angle C'BA = \angle CBA'$ and $\angle A'CB = \angle ACB'$. Let X, Y, Z be the feet of the perpendiculars from the points A', B', C' to the lines BC, CA, AB. Then, the lines AX, BY, CZ are concurrent.

Delta 51. (Cosmin Pohoață) Let ABC be a given triangle in plane. From each of its vertices we draw two arbitrary isogonals. Then, these six isogonals determine a hexagon with concurrent diagonals.

Epsilon 51. (USA 2003) Let ABC be a triangle. A circle passing through A and B intersects the segments AC and BC at D and E, respectively. Lines AB and DE intersect at F, while lines BD and CF intersect at M. Prove that $MF = MC$ if and only if $MB \cdot MD = MC^2$.

Delta 52. (Romanian jBMO 2007 Team Selection Test) Let ABC be a right triangle with $\angle A = 90°$, and let D be a point lying on the side AC. Denote by E reflection of A into the line BD, and by F the intersection point of CE with the perpendicular in D to the line BC. Prove that AF, DE and BC are concurrent.

Delta 53. Denote by AA_1, BB_1, CC_1 the altitudes of an acute triangle ABC, where A_1, B_1, C_1 lie on the sides BC, CA, and AB, respectively. A circle passing through A_1 and B_1 touches the arc AB of its circumcircle at C_2. The points A_2, B_2 are defined similarly.

1. (Tuymaada Olympiad 2007) Prove that the lines AA_2, BB_2, CC_2 are concurrent.
2. (Cosmin Pohoață, MathLinks Contest 2008, Round 1) Prove that the lines A_1A_2, B_1B_2, C_1C_2 are concurrent on the Euler line of ABC.

4.3. **Tossing onto Complex Plane.** Here, we discuss some applications of complex numbers to geometric inequality. Every complex number corresponds to a unique point in the complex plane. The standard symbol for the set of all complex numbers is $\mathbb{C}$, and we also refer to the complex plane as $\mathbb{C}$. We can identify the points in the real plane $\mathbb{R}^2$ as numbers in $\mathbb{C}$. The main tool is the following fundamental inequality.

Theorem 4.4. (Triangle Inequality) If $z_1, \cdots, z_n \in \mathbb{C}$, then $|z_1| + \cdots + |z_n| \geq |z_1 + \cdots + z_n|$.

Proof. Induction on n. □

Theorem 4.5. (Ptolemy's Inequality) For any points A, B, C, D in the plane, we have

$$AB \cdot CD + BC \cdot DA \geq AC \cdot BD.$$

Proof. Let a, b, c and 0 be complex numbers that correspond to A, B, C, D in the complex plane $\mathbb{C}$. It then becomes

$$|a - b| \cdot |c| + |b - c| \cdot |a| \geq |a - c| \cdot |b|.$$

Applying the Triangle Inequality to the identity $(a - b)c + (b - c)a = (a - c)b$, we get the result. □

Remark 4.1. *Investigate the equality case in Ptolemy's Inequality.*

Delta 54. [SL 1997 RUS] *Let $ABCDEF$ be a convex hexagon such that $AB = BC$, $CD = DE$, $EF = FA$. Prove that*

$$\frac{BC}{BE} + \frac{DE}{DA} + \frac{FA}{FC} \geq \frac{3}{2}.$$

When does the equality occur?

Epsilon 52. [TD] Let P be an arbitrary point in the plane of a triangle ABC with the centroid G. Show the following inequalities

(1) $BC \cdot PB \cdot PC + AB \cdot PA \cdot PB + CA \cdot PC \cdot PA \geq BC \cdot CA \cdot AB$,
(2) $PA^3 \cdot BC + PB^3 \cdot CA + PC^3 \cdot AB \geq 3PG \cdot BC \cdot CA \cdot AB$.

Delta 55. *Let H denote the orthocenter of an acute triangle ABC. Prove the geometric identity*

$$BC \cdot HB \cdot HC + AB \cdot HA \cdot HB + CA \cdot HC \cdot HA = BC \cdot CA \cdot AB.$$

Epsilon 53. (The Neuberg-Pedoe Inequality) Let a_1, b_1, c_1 denote the sides of the triangle $A_1B_1C_1$ with area F_1. Let a_2, b_2, c_2 denote the sides of the triangle $A_2B_2C_2$ with area F_2. Then, we have

$${a_1}^2({b_2}^2 + {c_2}^2 - {a_2}^2) + {b_1}^2({c_2}^2 + {a_2}^2 - {b_2}^2) + {c_1}^2({a_2}^2 + {b_2}^2 - {c_2}^2) \geq 16F_1F_2.$$

Epsilon 54. [SL 2002 KOR] *Let ABC be a triangle for which there exists an interior point F such that $\angle AFB = \angle BFC = \angle CFA$. Let the lines BF and CF meet the sides AC and AB at D and E, respectively. Prove that $AB + AC \geq 4DE$.*

4.4. **Generalize Ptolemy's Theorem!** The story begins with three trigonometric proofs of Ptolemy's Theorem.

Theorem 4.6. (Ptolemy's Theorem) Let $ABCD$ be a convex quadrilateral. If $ABCD$ is cyclic, then we have

$$AB \cdot CD + BC \cdot DA = AC \cdot BD.$$

First Proof. Set $AB = a$, $BC = b$, $CD = c$, $DA = d$. One natural approach is to compute $BD = x$ and $AC = y$ in terms of a, b, c and d. We apply the Cosine Law to obtain

$$x^2 = a^2 + d^2 - 2ad\cos A$$

and

$$x^2 = b^2 + c^2 - 2bc\cos D = b^2 + c^2 + 2bc\cos A.$$

Equating two equations, we meet

$$a^2 + d^2 - 2ad\cos A = b^2 + c^2 + 2bc\cos A$$

or

$$\cos A = \frac{a^2 + d^2 - b^2 - c^2}{2(ad + bc)}.$$

It follows that

$$x^2 = a^2 + d^2 - 2ad\cos A = a^2 + d^2 - 2ad\left(\frac{a^2 + d^2 - b^2 - c^2}{2(ad + bc)}\right) = \frac{(ac + bd)(ab + cd)}{ad + bc}.$$

Similarly, we also obtain

$$y^2 = \frac{(ac + bd)(ad + bc)}{ab + cd}.$$

Multiplying these two, we obtain $x^2y^2 = (ac + bd)^2$ or $xy = ac + bd$, as desired. □

Second Proof. (Hojoo Lee) As in the classical proof via the inversion, we rewrite it as

$$\frac{AB}{DA \cdot DB} + \frac{BC}{DB \cdot DC} = \frac{AC}{DA \cdot DC}.$$

We now trigonometrize each term. Letting $\mathcal{R}$ denote the circumradius of $ABCD$ and noticing that $\sin(\angle ADB) = \sin(\angle DBA + \angle DAB)$ in triangle DAB, we obtain

$$\begin{aligned}
\frac{AB}{DA \cdot DB} &= \frac{2\mathcal{R}\sin(\angle ADB)}{2\mathcal{R}\sin(\angle DBA) \cdot 2\mathcal{R}\sin(\angle DAB)} \\
&= \frac{\sin\angle DBA\cos\angle DAB + \cos\angle DBA\sin\angle DAB}{2\mathcal{R}\sin(\angle DBA)\sin(\angle DAB)} \\
&= \frac{1}{2\mathcal{R}}\left(\cot\angle DAB + \cot\angle DBA\right).
\end{aligned}$$

Likewise, we have

$$\frac{BC}{DB \cdot DC} = \frac{1}{2\mathcal{R}}\left(\cot\angle DBC + \cot\angle DCB\right)$$

and

$$\frac{AC}{DA \cdot DC} = \frac{1}{2\mathcal{R}}\left(\cot\angle DAC + \cot\angle DCA\right).$$

Hence, the geometric identity in Ptolemy's Theorem is equivalent to the cotangent identity

$$\left(\cot\angle DAB + \cot\angle DBA\right) + \left(\cot\angle DBC + \cot\angle DCB\right) = \left(\cot\angle DAC + \cot\angle DCA\right).$$

However, since the convex quadrilateral $ABCD$ admits a circumcircle, it is clear that

$$\angle DAB + \angle DCB = \pi, \ \angle DBA = \angle DCA, \ \angle DBC = \angle DAC$$

so that

$$\cot\angle DAB + \cot\angle DCB = 0, \ \cot\angle DBA = \cot\angle DCA, \ \cot\angle DBC = \cot\angle DAC.$$

□

Third Proof. We exploit the Sine Law to convert the geometric identity to the trigonometric identity. Let $\mathcal{R}$ denote the circumradius of $ABCD$. We set

$$\angle AOB = 2\theta_1,\ \angle BOC = 2\theta_2,\ \angle COD = 2\theta_3,\ \angle DOA = 2\theta_4,$$

where O is the center of the circumcircle of $ABCD$. It's clear that $\theta_1+\theta_2+\theta_3+\theta_4=\pi$. It follows that $AB = 2\mathcal{R}\sin\theta_1$, $BC = 2\mathcal{R}\sin\theta_2$, $CD = 2\mathcal{R}\sin\theta_3$, $DA = 2\mathcal{R}\sin\theta_4$, $AC = 2\mathcal{R}\sin(\theta_1+\theta_2)$, $AB = 2\mathcal{R}\sin(\theta_2+\theta_3)$. Our job is to establish

$$AB\cdot CD + BC\cdot DA = AC\cdot BD$$

or

$$(2\mathcal{R}\sin\theta_1)(2\mathcal{R}\sin\theta_3)+(2\mathcal{R}\sin\theta_2)(2\mathcal{R}\sin\theta_4)=(2\mathcal{R}\sin(\theta_1+\theta_2))(2\mathcal{R}\sin(\theta_2+\theta_3))$$

or equivalently

$$\sin\theta_1\sin\theta_3+\sin\theta_2\sin\theta_4=\sin(\theta_1+\theta_2)\sin(\theta_2+\theta_3).$$

We use the well-known identity $\sin\alpha\sin\beta=\frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right]$ to rewrite it as

$$\begin{aligned}&\frac{\cos(\theta_1-\theta_3)-\cos(\theta_1+\theta_3)}{2}+\frac{\cos(\theta_2-\theta_4)-\cos(\theta_2+\theta_4)}{2}\\ =\ &\frac{\cos(\theta_1-\theta_3)-\cos(\theta_1+2\theta_2+\theta_3)}{2}\end{aligned}$$

or equivalently

$$-\cos(\theta_1+\theta_3)+\cos(\theta_2-\theta_4)-\cos(\theta_2+\theta_4)=-\cos(\theta_1+2\theta_2+\theta_3).$$

Since $\theta_1+\theta_2+\theta_3+\theta_4=\pi$ or since $\cos(\theta_1+\theta_3)+\cos(\theta_2+\theta_4)=0$, it is equivalent to

$$\cos(\theta_2-\theta_4)=-\cos(\theta_1+2\theta_2+\theta_3).$$

However, we employ $\theta_1+\theta_2+\theta_3=\pi-\theta_4$ to deduce

$$\cos(\theta_1+2\theta_2+\theta_3)=\cos(\theta_2+\pi-\theta_4)=-\cos(\theta_2-\theta_4).$$

□

When the second author of this weblication was a high school student, one day, he was trying to device a coordinate proof of Ptolemy's Theorem. However, we immediately realize that the direct approach using only the distance formula is hopeless. The geometric identity reads, in coordinates,

$$\begin{aligned}&\sqrt{\left\{(x_1-x_2)^2+(y_1-y_2)^2\right\}\left\{(x_3-x_4)^2+(y_3-y_4)^2\right\}}\\ &+\sqrt{\left\{(x_2-x_3)^2+(y_2-y_3)^2\right\}\left\{(x_4-x_1)^2+(y_4-y_1)^2\right\}}\\ &=\sqrt{\left\{(x_1-x_3)^2+(y_1-y_3)^2\right\}\left\{(x_2-x_4)^2+(y_2-y_4)^2\right\}}.\end{aligned}$$

What he realized was that the key point is to find a natural coordinate condition. First, forget about the destination $AB\cdot CD+BC\cdot DA=AC\cdot BD$ and, instead, find out what the initial condition that $ABCD$ is cyclic says in coordinates.

Lemma 4.1. *Let $ABCD$ be a convex quadrilateral. We toss $ABCD$ on the real plane $\mathbb{R}^2$ with the coordinates $A(a_1,a_2)$, $B(b_1,b_2)$, $C(c_1,c_2)$, $D(d_1,d_2)$. Then, the necessary and sufficient condition that $ABCD$ is cyclic is that the following equality holds.*

$$\begin{aligned}&\frac{{a_1}^2+{a_2}^2-(a_1b_1+a_2b_2+a_1c_1+a_2c_2-b_1c_1-b_2c_2)}{b_1a_2+a_1c_2+c_1b_2-a_1b_2-c_1a_2-b_1c_2}\\ =\ &\frac{{d_1}^2+{d_2}^2-(d_1b_1+d_2b_2+d_1a_1+d_2a_2-b_1a_1-b_2a_2)}{b_1d_2+d_1a_2+a_1b_2-d_1b_2-a_1d_2-b_1a_2}\end{aligned}$$

Proof. The quadrilateral $ABCD$ is cyclic if and only if $\angle BAC = \angle BDC$, or equivalently $\cot(\angle BAC) = \cot(\angle BDC)$. It is equivalent to

$$\frac{\cos(\angle BAC)}{\sin(\angle BAC)} = \frac{\cos(\angle BDC)}{\sin(\angle BDC)}$$

or

$$\frac{\frac{BA^2+AC^2-CB^2}{2BA\cdot AC}}{\frac{2[ABC]}{BA\cdot AC}} = \frac{\frac{BD^2+DC^2-CB^2}{2BD\cdot DC}}{\frac{2[DBC]}{BD\cdot DC}}$$

or

$$\frac{BA^2+AC^2-CB^2}{2[ABC]} = \frac{BD^2+DC^2-CB^2}{2[DBC]}$$

or in coordinates,

$$\begin{aligned} &\frac{{a_1}^2+{a_2}^2-(a_1b_1+a_2b_2+a_1c_1+a_2c_2-b_1c_1-b_2c_2)}{b_1a_2+a_1c_2+c_1b_2-a_1b_2-c_1a_2-b_1c_2} \\ = &\frac{{d_1}^2+{d_2}^2-(d_1b_1+d_2b_2+d_1a_1+d_2a_2-b_1a_1-b_2a_2)}{b_1d_2+d_1a_2+a_1b_2-d_1b_2-a_1d_2-b_1a_2}. \end{aligned}$$

□

The coordinate condition and its proof is natural. However, something weird happens here. It does not look like being cyclic in the coordinates . Indeed, when $ABCD$ is a cyclic quadrilateral, we notice that the same quadrilateral $BCDA$, $CDAB$, $DABC$ are also trivially cyclic. It turns out that the coordinate condition indeed admits a certain symmetry. Now, it is time to consider the destination

$$AB\cdot CD + BC\cdot DA = AC\cdot BD.$$

As we see above, the direct application of the distance formula gives us a monster identity with square roots. What we need is a reformulation without square roots. We recall that Ptolemy's Theorem is trivialized by the inversive geometry. As in the proof via the inversion, we rewrite it in the symmetric form

$$\frac{AB}{DA\cdot DB} + \frac{BC}{DB\cdot DC} = \frac{AC}{DA\cdot DC}.$$

Now, we reach the key step. Let $\mathcal{R}$ denote the circumcircle of $ABCD$. The formulas

$$[DAB] = \frac{AB\cdot DA\cdot DB}{4\mathcal{R}},\ [DBC] = \frac{BC\cdot DB\cdot DC}{4\mathcal{R}},\ [DCA] = \frac{CA\cdot DC\cdot DA}{4\mathcal{R}}$$

allows us to realize that it is equivalent to the geometric identity.

$$\frac{[DAB]}{DA^2\cdot DB^2} + \frac{[DBC]}{DB^2\cdot DC^2} = \frac{[DAC]}{DA^2\cdot DC^2}$$

or

$$DC^2[DAB] + DA^2[DBC] = DB^2[DAC].$$

Summarizing up the result, we have

Lemma 4.2. *Let $ABCD$ be a convex and cyclic quadrilateral. Then the following two geometric identities are equivalent.*

(1) $AB\cdot CD + BC\cdot DA = AC\cdot BD$.

(2) $DC^2[DAB] + DA^2[DBC] = DB^2[DAC]$.

It is awesome. Why? It is because we can express the second condition in the coordinates without the horrible square root! After a long, very long computation by hand, we can check that

$$\begin{aligned} &\frac{{a_1}^2+{a_2}^2-(a_1b_1+a_2b_2+a_1c_1+a_2c_2-b_1c_1-b_2c_2)}{b_1a_2+a_1c_2+c_1b_2-a_1b_2-c_1a_2-b_1c_2} \\ = &\frac{{d_1}^2+{d_2}^2-(d_1b_1+d_2b_2+d_1a_1+d_2a_2-b_1a_1-b_2a_2)}{b_1d_2+d_1a_2+a_1b_2-d_1b_2-a_1d_2-b_1a_2}. \end{aligned}$$

indeed implies the coordinate condition of the reformulation (2). The brute-force computation will be simplified if we take $D(d_1, d_2) = (0, 0)$. However, it is not the end of the story. Actually, he found a symmetry in the coordinate computation. It leads him to rediscover The Feuerbach-Luchterhand Theorem, which generalize Ptolemy's Theorem.

Theorem 4.7. (The Feuerbach-Luchterhand Theorem) Let $ABCD$ be a convex and cyclic quadrilateral. For any point O in the plane, we have

$$OA^2 \cdot BC \cdot CD \cdot DB - OB^2 \cdot CD \cdot DA \cdot AB + OC^2 \cdot DA \cdot AB \cdot BD - OD^2 \cdot AB \cdot BC \cdot CD = 0.$$

Proof. We toss the picture on the real plane $\mathbb{R}^2$ with the coordinates

$$O(0,0),\ A(a_1, a_2),\ B(b_1, b_2),\ C(c_1, c_2),\ D(d_1, d_2),$$

Letting $\mathcal{R}$ denote the circumcircle of $ABCD$, it can be rewritten as

$$OA^2 \cdot \frac{[BCD]}{4R} - OB^2 \cdot \frac{[CDA]}{4R} + OC^2 \cdot \frac{[DAB]}{4R} - OD^2 \cdot \frac{[DBC]}{4R} = 0$$

or

$$OA^2 \cdot [BCD] - OB^2 \cdot [CDA] + OC^2 \cdot [DAB] - OD^2 \cdot [ABC] = 0.$$

We can rewrite this in the coordinates without square roots. Now, after long computation, we can check, by hand, that it is equivalent to the coordinate condition that $ABCD$ is cyclic:

$$\begin{aligned} & \frac{{a_1}^2 + {a_2}^2 - (a_1b_1 + a_2b_2 + a_1c_1 + a_2c_2 - b_1c_1 - b_2c_2)}{b_1a_2 + a_1c_2 + c_1b_2 - a_1b_2 - c_1a_2 - b_1c_2} \\ = & \frac{{d_1}^2 + {d_2}^2 - (d_1b_1 + d_2b_2 + d_1a_1 + d_2a_2 - b_1a_1 - b_2a_2)}{b_1d_2 + d_1a_2 + a_1b_2 - d_1b_2 - a_1d_2 - b_1a_2}. \end{aligned}$$

□

The end? Not yet. It turns out that after throwing out the essential condition that the quadrilateral is cyclic, we can extend the theorem to arbitrary quadrilaterals!

Theorem 4.8. (Hojoo Lee) For an arbitrary point P in the plane of the convex quadrilateral $A_1A_2A_3A_4$, we obtain

$$\begin{aligned} & {PA_1}^2[\triangle A_2A_3A_4] - {PA_2}^2[\triangle A_3A_4A_1] + {PA_3}^2[\triangle A_4A_1A_2] - {PA_4}^2[\triangle A_1A_2A_3] \\ = \ & \overrightarrow{A_1A_2} \cdot \overrightarrow{A_1A_3}\,[\triangle A_2A_3A_4] - \overrightarrow{A_4A_2} \cdot \overrightarrow{A_4A_3}\,[\triangle A_1A_2A_3]. \end{aligned}$$

After removing the convexity of $A_1A_2A_3A_4$, we get the same result regarding the signed area of triangle.

Outline of Proof. We toss the figure on the real plane $\mathbb{R}^2$ and write $P(0,0)$ and $A_i = (x_i, y_i)$, where $1 \le i \le 4$. Our task is to check that two matrices

$$L = \begin{pmatrix} {PA_1}^2 & {PA_2}^2 & {PA_3}^2 & {PA_4}^2 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

and

$$R = \begin{pmatrix} \overrightarrow{A_1A_2} \cdot \overrightarrow{A_1A_3} & 0 & 0 & \overrightarrow{A_4A_2} \cdot \overrightarrow{A_4A_3} \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

have the same determinant. We now invite the readers to find a neat proof, of course without the brute force expansion of the determinants! □

Is it a re-discovery again? Is this the end of the story? No. There is no end of generalizations in Mathematics. The lesson we want to deliver here is simple: Even the brute-force coordinate proofs offer good motivations. There is no bad proof. We now present some applications of the The Feuerbach-Luchterhand Theorem.

Corollary 4.2. *Let $ABCD$ be a rectangle. For any point P, we have*

$$PA^2 - PB^2 + PC^2 - PD^2 = 0.$$

Now, let's see what happens if we apply The Feuerbach-Luchterhand Theorem to a geometric situation from the triangle geometry. Let ABC be a triangle with the incenter I and the circumcenter O. Let $BC = a$, $CA = b$, $AB = c$, $s = \frac{a+b+c}{2}$. Let R and r denote the circumradius and inradius, respectively. Let P and Q denote the feet of perpendiculars from I to the sides CA and CB, respectively. Since $\angle IPC = 90^\circ = \angle IQC$, we find that $IQCP$ is cyclic.

We then apply The Feuerbach-Luchterhand Theorem to the pair $(O, IQCP)$ to deduce the geometric identity

$$0 = OI^2QC \cdot CP \cdot PQ - OQ^2CP \cdot PI \cdot IC + OC^2PI \cdot IQ \cdot QP - OP^2IQ \cdot QC \cdot CI.$$

What does it mean? We observe that, in the isosceles triangles COA and BOC,

$$OP^2 = R^2 - AP \cdot PC = R^2 - (s-a)(s-c),$$
$$OQ^2 = R^2 - BQ \cdot QC = R^2 - (s-b)(s-c).$$

Now, it becomes

$$\begin{aligned} 0 &= OI^2(s-c)^2\left(CI\frac{c}{2R}\right) - \left[R^2-(s-b)(s-c)\right](s-c)r \cdot IC \\ &+ R^2r^2\left(CI\frac{c}{2R}\right) - \left[R^2-(s-a)(s-c)\right]r(s-c)\cdot CI \end{aligned}$$

or

$$\begin{aligned} 0 &= OI^2(s-c)^2 \cdot \frac{c}{2R} - \left[R^2-(s-b)(s-c)\right](s-c)r \\ &+ R^2r^2 \cdot \frac{c}{2R} - \left[R^2-(s-a)(s-c)\right]r(s-c). \end{aligned}$$

or

$$OI^2(s-c)^2 \cdot \frac{c}{2R} = -\frac{Rr^2c}{2} + \left[2R^2 - c(s-c)\right](s-c)r$$

or

$$\frac{OI^2}{R} = -\frac{Rr^2}{(s-c)^2} + \frac{4R^2r}{c(s-c)} - 2r = R\left[\frac{4Rr(s-c)-r^2c}{(s-c)^2c}\right] - 2r.$$

Now, we apply Ptolemy's Theorem and The Pythagoras Theorem to deduce

$$2r(s-c) = CP \cdot IQ + PI \cdot IQ = PQ \cdot CI = CI^2\frac{c}{2R} = \left[(s-c)^2+r^2\right]\frac{c}{2R}$$

or

$$4Rr(s-c) = c\left((s-c)^2+r^2\right)$$

or

$$4Rr(s-c) - r^2c = (s-c)^2c$$

or

$$\frac{4Rr(s-c)-r^2c}{(s-c)^2c} = 1.$$

It therefore follows that

$$OI^2 = R^2 - 2rR.$$

It is the theorem proved first by L. Euler. There are lots of way to establish this. Device your own proofs! Find other corollaries of The Feuerbach-Luchterhand Theorem. Another possible generalization of Ptolemy's relation is Casey's theorem:

Theorem 4.9. (Casey's Theorem) Given four circles $\mathcal{C}_i$, $i = 1,\ 2,\ 3,\ 4$, let t_{ij} be the length of a common tangent between $\mathcal{C}_i$ and $\mathcal{C}_j$. The four circles are tangent to a fifth circle (or line) if and only if for an appropriate choice of signs, we have that

$$t_{12}t_{34} \pm t_{13}t_{42} \pm t_{14}t_{23} = 0.$$

The most common proof for this result is by making use of inversion. See [RJ]. We shall omit it here. We now work on the Feuerbach's celebrated theorem (actually its first version).

Theorem 4.10. (Feuerbach's Theorem) The incircle and nine-point circle of a triangle are tangent to one another.

Why *first* version? Of course, most of you might know that the nine-point circle is also tangent to the three excircles of the triangle. Most of the geometry textbooks include this last remark in the theorem's statement as well, but this is mostly for sake of completeness, since the proof is similar with the incenter case.

Proof. Let the sides BC, CA, AB of triangle ABC have midpoints D, E, F respectively, and let Γ be the incircle of the triangle. Let a, b, c be the sidelengths of ABC, and let s be its semiperimeter. We now consider the 4-tuple of circles $(D,\ E,\ F,\ \Gamma)$. Here is what we find:

$$t_{DE} = \frac{c}{2},\quad t_{DF} = \frac{b}{2},\quad t_{EF} = \frac{a}{2},$$

$$t_{D\Gamma} = \left|\frac{a}{2} - (s-b)\right| = \left|\frac{b-c}{2}\right|,$$

$$t_{E\Gamma} = \left|\frac{b}{2} - (s-c)\right| = \left|\frac{a-c}{2}\right|,$$

$$t_{F\Gamma} = \left|\frac{c}{2} - (s-a)\right| = \left|\frac{b-a}{2}\right|.$$

We need to check whether, for some combination of $+,\ -$ signs, we have

$$\pm(c(b-a) \pm a(b-c) \pm b(a-c) = 0.$$

But this is immediate! According to Casey's theorem there exists a circle what touches each of D, E, F and Γ. Since the circle passing through D, E, F is the ninepoint circle of the triangle, it follows that Γ and the nine-point circle are tangent to each other. □

We shall see now an interesting particular case of Thebault's theorem.

Proposition 4.2. (IMO Longlist 1991, proposed by India) Circles Γ_1 and Γ_2 are externally tangent at a point I, and both are enclosed by and tangent to a third circle Γ. One common tangent to Γ_1 and Γ_2 meets Γ in B and C, while the common tangent at I meets Γ in A on the same side of BC as I. Then, we have that I is the incenter of triangle ABC.

Proof. Let X, Y be the tangency points of BC with the circles Γ_1, and Γ_2, respectively, and let x, y be the lengths of the tangents from B and C to Γ_1 and Γ_2. Denote by D the intersection of AI with the line BC, and let $z = AI$, $u = ID$. According to Casey's theorem, applied for the two 4-tuples of circles $(A,\ \Gamma_1,\ B,\ C)$ and $(A,\ \Gamma_2,\ C,\ B)$, we obtain $az + bx = c(2u + y)$ and $az + cy = b(2u + x)$. Subtracting the second equation from the first, we obtain that $bx - cy = u(c - b)$, and therefore $\frac{x+u}{y+u} = \frac{c}{b}$, that is, $\frac{BD}{DC} = \frac{AB}{AC}$, which implies that AI bisects $\angle BAC$, and that $BD = \frac{ac}{b+c}$. Adding the two equations mentioned before, we finally obtain that $az = u(b + c)$, which rewrites as $\frac{AI}{ID} = \frac{AB}{BD}$. This implies that BI bisects $\angle ABC$. Thus, I is the incenter of triangle ABC. □

But can we prove Thebault's theorem using Casey? S. Gueron [SG] says yes!

Delta 56. (Thebault [VT]) Through the vertex A of a triangle ABC, a straight line AD is drawn, cutting the side BC at D. I is the incenter of triangle ABC, and let P be the center of the circle which touches DC, DA, and (internally) the circumcircle of ABC, and let Q be the center of the circle which touches DB, DA, and (internally) the circumcircle of ABC. Then, the points P, I, Q are collinear.

Delta 57. (Jean-Pierre Ehrmann and Cosmin Pohoață, MathLinks Contest 2008) Let P be an arbitrary point on the side BC of a given triangle ABC with circumcircle Γ. Let $\mathcal{T}_A^b$ be the circle tangent to AP, PB, and internally to Γ, and let $\mathcal{T}_A^c$ be the circle tangent to AP, PC, and internally to Γ. Then, the circles $\mathcal{T}_A^b$ and $\mathcal{T}_A^c$ are congruent if and only if AP passes through the Nagel point of triangle ABC.

Delta 58. (Lev Emelyanov [LE]) Let P be a point in the interior of a given triangle ABC. Denote by A_1, B_1, C_1 the intersections of AP, BP, CP with the sidelines BC, CA, and AB, respectively (in other words, the triangle $A_1B_1C_1$ is the cevian triangle of P with respect to ABC). Construct the three circles (O_1), (O_2) and (O_3) outside the triangle which are tangent to the sides of ABC at A_1, B_1, C_1, and also tangent to the circumcircle of ABC. Then, the circle tangent externally to these three circles is also tangent to teh incircle of triangle ABC.

It is impossible to be a mathematician without being a poet in soul. - S. Kovalevskaya

5. Three Terrific Techniques (EAT)

A long time ago an older and well-known number theorist made some disparaging remarks about Paul Erdős's work. You admire Erdős's contributions to mathematics as much as I do, and I felt annoyed when the older mathematician flatly and definitively stated that all of Erdős's work could be "reduced" to a few tricks which Erdős repeatedly relied on in his proofs. What the number theorist did not realize is that other mathematicians, even the very best, also rely on a few tricks which they use over and over. **Take Hilbert.** The second volume of Hilbert's collected papers contains Hilbert's papers in invariant theory. I have made a point of reading some of these papers with care. It is sad to note that some of Hilbert's beautiful results have been completely forgotten. But on reading the proofs of Hilbert's striking and deep theorems in invariant theory, it was surprising to verify that Hilbert's proofs relied on the same few tricks. **Even Hilbert had only a few tricks!**

- G-C Rota, *Ten Lessons I Wish I Had Been Taught*

5.1. **'T'rigonometric Substitutions.** If you are faced with an integral that contains square root expressions such as

$$\int \sqrt{1-x^2}\,dx, \quad \int \sqrt{1+y^2}\,dy, \quad \int \sqrt{z^2-1}\,dz$$

then trigonometric substitutions such as $x = \sin t$, $y = \tan t$, $z = \sec t$ are very useful. We will learn that making a suitable trigonometric substitution simplifies the given inequality.

Epsilon 55. (APMO 2004/5) Prove that, for all positive real numbers a, b, c,

$$(a^2+2)(b^2+2)(c^2+2) \geq 9(ab+bc+ca).$$

Epsilon 56. (Latvia 2002) Let a, b, c, d be the positive real numbers such that

$$\frac{1}{1+a^4}+\frac{1}{1+b^4}+\frac{1}{1+c^4}+\frac{1}{1+d^4}=1.$$

Prove that $abcd \geq 3$.

Epsilon 57. (Korea 1998) Let x, y, z be the positive reals with $x+y+z = xyz$. Show that

$$\frac{1}{\sqrt{1+x^2}}+\frac{1}{\sqrt{1+y^2}}+\frac{1}{\sqrt{1+z^2}} \leq \frac{3}{2}.$$

Since the function $f(t) = \frac{1}{\sqrt{1+t^2}}$ is not concave on $\mathbb{R}^+$, we cannot apply Jensen's Inequality directly. However, the function $f(\tan\theta)$ is concave on $\left(0, \frac{\pi}{2}\right)$!

Proposition 5.1. *In any acute triangle ABC, we have* $\cos A + \cos B + \cos C \leq \frac{3}{2}$.

Proof. Since $\cos x$ is concave on $\left(0, \frac{\pi}{2}\right)$, it's a direct consequence of Jensen's Inequality. □

We note that the function $\cos x$ is not concave on $(0, \pi)$. In fact, it's convex on $\left(\frac{\pi}{2}, \pi\right)$. One may think that the inequality $\cos A+\cos B+\cos C \leq \frac{3}{2}$ doesn't hold for any triangles. However, it's known that it holds for all triangles.

Proposition 5.2. *In any triangle ABC, we have*

$$\cos A + \cos B + \cos C \leq \frac{3}{2}.$$

First Proof. It follows from $\pi - C = A + B$ that

$$\cos C = -\cos(A+B) = -\cos A \cos B + \sin A \sin B$$

or

$$3 - 2(\cos A + \cos B + \cos C) = (\sin A - \sin B)^2 + (\cos A + \cos B - 1)^2 \geq 0.$$

$\square$

Second Proof. Let $BC = a$, $CA = b$, $AB = c$. Use The Cosine Law to rewrite the given inequality in the terms of a, b, c :

$$\frac{b^2+c^2-a^2}{2bc} + \frac{c^2+a^2-b^2}{2ca} + \frac{a^2+b^2-c^2}{2ab} \leq \frac{3}{2}.$$

Clearing denominators, this becomes

$$3abc \geq a(b^2+c^2-a^2) + b(c^2+a^2-b^2) + c(a^2+b^2-c^2),$$

which is equivalent to $abc \geq (b+c-a)(c+a-b)(a+b-c)$. $\square$

We remind that the geometric inequality $R \geq 2r$ is equivalent to the *algebraic* inequality $abc \geq (b+c-a)(c+a-b)(a+b-c)$. We now find that, in the proof of the above theorem, $abc \geq (b+c-a)(c+a-b)(a+b-c)$ is equivalent to the trigonometric inequality $\cos A + \cos B + \cos C \leq \frac{3}{2}$. One may ask that

> in any triangles ABC, is there a natural relation between $\cos A + \cos B + \cos C$ and $\frac{R}{r}$, where R and r are the radii of the circumcircle and incircle of ABC?

Theorem 5.1. *Let R and r denote the radii of the circumcircle and incircle of the triangle ABC. Then, we have*

$$\cos A + \cos B + \cos C = 1 + \frac{r}{R}.$$

Proof. Use the algebraic identity

$$a(b^2+c^2-a^2) + b(c^2+a^2-b^2) + c(a^2+b^2-c^2) = 2abc + (b+c-a)(c+a-b)(a+b-c).$$

We leave the details for the readers. $\square$

Delta 59. (China 2004) Let ABC be a triangle with $BC = a, CA = b, AB = c$. Prove that, for all $x \geq 0$,

$$a^x \cos A + b^x \cos B + c^x \cos C \leq \frac{1}{2}\left(a^x + b^x + c^x\right).$$

Delta 60. *(a) Let p, q, r be the positive real numbers such that $p^2 + q^2 + r^2 + 2pqr = 1$. Show that there exists an acute triangle ABC such that $p = \cos A$, $q = \cos B$, $r = \cos C$.*
(b) Let $p, q, r \geq 0$ with $p^2 + q^2 + r^2 + 2pqr = 1$. Show that there are $A, B, C \in \left[0, \frac{\pi}{2}\right]$ with $p = \cos A$, $q = \cos B$, $r = \cos C$, and $A + B + C = \pi$.

Epsilon 58. (USA 2001) Let a, b, and c be nonnegative real numbers such that $a^2 + b^2 + c^2 + abc = 4$. Prove that $0 \leq ab + bc + ca - abc \leq 2$.

Life is good for only two things, discovering mathematics and teaching mathematics.
- S. Poisson

5.2. **'A'lgebraic Substitutions.** We know that some inequalities in triangle geometry can be treated by the Ravi substitution and trigonometric substitutions. We can also transform the given inequalities into easier ones through some clever algebraic substitutions.

Epsilon 59. [IMO 2001/2 KOR] *Let a, b, c be positive real numbers. Prove that*
$$\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ca}}+\frac{c}{\sqrt{c^2+8ab}}\geq 1.$$

Epsilon 60. [IMO 1995/2 RUS] *Let a, b, c be positive numbers such that $abc = 1$. Prove that*
$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)}\geq\frac{3}{2}.$$

Epsilon 61. (Korea 1998) Let x, y, z be the positive reals with $x+y+z=xyz$. Show that
$$\frac{1}{\sqrt{1+x^2}}+\frac{1}{\sqrt{1+y^2}}+\frac{1}{\sqrt{1+z^2}}\leq\frac{3}{2}.$$

We now prove a classical theorem in various ways.

Proposition 5.3. (Nesbitt) For all positive real numbers a, b, c, we have
$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}.$$

Proof 4. *After the substitution $x=b+c$, $y=c+a$, $z=a+b$, it becomes*
$$\sum_{\text{cyclic}}\frac{y+z-x}{2x}\geq\frac{3}{2}\quad or\quad\sum_{\text{cyclic}}\frac{y+z}{x}\geq 6,$$
which follows from The AM-GM Inequality as following:
$$\sum_{\text{cyclic}}\frac{y+z}{x}=\frac{y}{x}+\frac{z}{x}+\frac{z}{y}+\frac{x}{y}+\frac{x}{z}+\frac{y}{z}\geq 6\left(\frac{y}{x}\cdot\frac{z}{x}\cdot\frac{z}{y}\cdot\frac{x}{y}\cdot\frac{x}{z}\cdot\frac{y}{z}\right)^{\frac{1}{6}}=6.$$

Proof 5. *We make the substitution*
$$x=\frac{a}{b+c},\ y=\frac{b}{c+a},\ z=\frac{c}{a+b}.$$
It follows that
$$\sum_{\text{cyclic}}f(x)=\sum_{\text{cyclic}}\frac{a}{a+b+c}=1,$$
where $f(t)=\frac{t}{1+t}$. Since f is concave on $(0,\infty)$, Jensen's Inequality shows that
$$f\left(\frac{1}{2}\right)=\frac{1}{3}=\frac{1}{3}\sum_{\text{cyclic}}f(x)\leq f\left(\frac{x+y+z}{3}\right)$$
Since f is monotone increasing, it implies that
$$\frac{1}{2}\leq\frac{x+y+z}{3}$$
or
$$\sum_{\text{cyclic}}\frac{a}{b+c}=x+y+z\geq\frac{3}{2}.$$

Proof 6. *As in the previous proof, it suffices to show that*
$$T:=\frac{x+y+z}{3}\geq\frac{1}{2},$$
where we have
$$\sum_{\text{cyclic}}\frac{x}{1+x}=1$$

or equivalently,

$$1 = 2xyz + xy + yz + zx.$$

We apply The AM-GM Inequality to deduce

$$1 = 2xyz + xy + yz + zx \leq 2T^3 + 3T^2$$

It follows that

$$2T^3 + 3T^2 - 1 \geq 0$$

so that

$$(2T - 1)(T + 1)^2 \geq 0$$

or

$$T \geq \frac{1}{2}.$$

Epsilon 62. [IMO 2000/2 USA] *Let a, b, c be positive numbers such that $abc = 1$. Prove that*

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1.$$

Epsilon 63. *Let a, b, c be positive real numbers satisfying $a + b + c = 1$. Show that*

$$\frac{a}{a + bc} + \frac{b}{b + ca} + \frac{\sqrt{abc}}{c + ab} \leq 1 + \frac{3\sqrt{3}}{4}.$$

Epsilon 64. (Latvia 2002) Let a, b, c, d be the positive real numbers such that

$$\frac{1}{1 + a^4} + \frac{1}{1 + b^4} + \frac{1}{1 + c^4} + \frac{1}{1 + d^4} = 1.$$

Prove that $abcd \geq 3$.

Delta 61. [SL 1993 USA] *Prove that*

$$\frac{a}{b + 2c + 3d} + \frac{b}{c + 2d + 3a} + \frac{c}{d + 2a + 3b} + \frac{d}{a + 2b + 3c} \geq \frac{2}{3}$$

for all positive real numbers a, b, c, d.

Epsilon 65. [LL 1992 UNK] (Iran 1998) Prove that, for all $x, y, z > 1$ such that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2$,

$$\sqrt{x + y + z} \geq \sqrt{x - 1} + \sqrt{y - 1} + \sqrt{z - 1}.$$

Epsilon 66. (Belarus 1998) Prove that, for all $a, b, c > 0$,

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{a + b}{b + c} + \frac{b + c}{c + a} + 1.$$

Delta 62. [IMO 1969 USS] *Under the conditions $x_1, x_2 > 0$, $x_1y_1 > {z_1}^2$, and $x_2y_2 > {z_2}^2$, prove the inequality*

$$\frac{8}{(x_1 + x_2)(y_1 + y_2) - (z_1 + z_2)^2} \leq \frac{1}{x_1y_1 - {z_1}^2} + \frac{1}{x_2y_2 - {z_2}^2}.$$

Epsilon 67. [SL 2001] *Let $x_1, \cdots, x_n$ be arbitrary real numbers. Prove the inequality.*

$$\frac{x_1}{1 + {x_1}^2} + \frac{x_2}{1 + {x_1}^2 + {x_2}^2} + \cdots + \frac{x_n}{1 + {x_1}^2 + \cdots + {x_n}^2} < \sqrt{n}.$$

Delta 63. [LL 1987 FRA] *Given n real numbers $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n < 1$, prove that*

$$\left(1 - {t_n}^2\right)\left(\frac{t_1}{(1 - {t_1}^2)^2} + \frac{{t_2}^2}{(1 - {t_2}^3)^2} + \cdots + \frac{{t_n}^n}{(1 - {t_n}^{n+1})^2}\right) < 1.$$

5.3. **'E'stablishing New Bounds.** The following examples give a nice description of the title of this subsection.

Example 10. *Let x, y, z be positive real numbers. Show the cyclic inequality*

$$\frac{x^2}{y^2}+\frac{y^2}{z^2}+\frac{z^2}{x^2} \geq \frac{x}{y}+\frac{y}{z}+\frac{z}{x}.$$

Second Solution. We first use the auxiliary inequality $t^2 \geq 2t-1$ to deduce

$$\frac{x^2}{y^2}+\frac{y^2}{z^2}+\frac{z^2}{x^2} \geq 2\frac{x}{y}-1+2\frac{y}{z}-1+2\frac{z}{x}-1.$$

It now remains to check that

$$2\frac{x}{y}-1+2\frac{y}{z}-1+2\frac{z}{x}-1 \geq \frac{x}{y}+\frac{y}{z}+\frac{z}{x}$$

or equivalently

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x} \geq 3.$$

However, The AM-GM Inequality shows that

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x} \geq 3\left(\frac{x}{y}\cdot\frac{y}{z}\cdot\frac{z}{x}\right)^{\frac{1}{3}}=3.$$

$\square$

Proposition 5.4. (Nesbitt) For all positive real numbers a, b, c, we have

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq \frac{3}{2}.$$

Proof 7. *From* $\left(\frac{a}{b+c}-\frac{1}{2}\right)^2 \geq 0$*, we deduce that*

$$\frac{a}{b+c} \geq \frac{1}{4}\cdot\frac{\frac{8a}{b+c}-1}{\frac{a}{b+c}+1}=\frac{8a-b-c}{4(a+b+c)}.$$

It follows that

$$\sum_{\text{cyclic}}\frac{a}{b+c} \geq \sum_{\text{cyclic}}\frac{8a-b-c}{4(a+b+c)}=\frac{3}{2}.$$

Proof 8. *We claim that*

$$\frac{a}{b+c} \geq \frac{3a^{\frac{3}{2}}}{2\left(a^{\frac{3}{2}}+b^{\frac{3}{2}}+c^{\frac{3}{2}}\right)} \quad \text{or} \quad 2\left(a^{\frac{3}{2}}+b^{\frac{3}{2}}+c^{\frac{3}{2}}\right) \geq 3a^{\frac{1}{2}}(b+c).$$

The AM-GM inequality gives $a^{\frac{3}{2}}+b^{\frac{3}{2}}+b^{\frac{3}{2}} \geq 3a^{\frac{1}{2}}b$ *and* $a^{\frac{3}{2}}+c^{\frac{3}{2}}+c^{\frac{3}{2}} \geq 3a^{\frac{1}{2}}c$ *. Adding these two inequalities yields* $2\left(a^{\frac{3}{2}}+b^{\frac{3}{2}}+c^{\frac{3}{2}}\right) \geq 3a^{\frac{1}{2}}(b+c)$*, as desired. Therefore, we have*

$$\sum_{\text{cyclic}}\frac{a}{b+c} \geq \frac{3}{2}\sum_{\text{cyclic}}\frac{a^{\frac{3}{2}}}{a^{\frac{3}{2}}+b^{\frac{3}{2}}+c^{\frac{3}{2}}}=\frac{3}{2}.$$

Epsilon 68. *Let a, b, c be the lengths of a triangle. Show that*

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} < 2.$$

Some cyclic inequalities can be established by finding some clever bounds. Suppose that we want to establish that

$$\sum_{\text{cyclic}} F(x,y,z) \geq C$$

for some given constant $C \in \mathbb{R}$. Whenever we have a function G such that, for all $x, y, z > 0$,

$$F(x,y,z) \geq G(x,y,z)$$

and

$$\sum_{\text{cyclic}} G(x,y,z) = C,$$

we then deduce that

$$\sum_{\text{cyclic}} F(x,y,z) \geq \sum_{\text{cyclic}} G(x,y,z) = C.$$

For instance, if a function F satisfies the inequality

$$F(x,y,z) \geq \frac{x}{x+y+z}$$

for all $x, y, z > 0$, then F obeys the inequality

$$\sum_{\text{cyclic}} F(x,y,z) \geq 1.$$

Epsilon 69. [IMO 2001/2 KOR] *Let a, b, c be positive real numbers. Prove that*

$$\frac{a}{\sqrt{a^2+8bc}} + \frac{b}{\sqrt{b^2+8ca}} + \frac{c}{\sqrt{c^2+8ab}} \geq 1.$$

Epsilon 70. [IMO 2005/3 KOR] *Let x, y, and z be positive numbers such that $xyz \geq 1$. Prove that*

$$\frac{x^5-x^2}{x^5+y^2+z^2} + \frac{y^5-y^2}{y^5+z^2+x^2} + \frac{z^5-z^2}{z^5+x^2+y^2} \geq 0.$$

Epsilon 71. (KMO Weekend Program 2007) Prove that, for all $a, b, c, x, y, z > 0$,

$$\frac{ax}{a+x} + \frac{by}{b+y} + \frac{cz}{c+z} \leq \frac{(a+b+c)(x+y+z)}{a+b+c+x+y+z}.$$

Epsilon 72. (USAMO Summer Program 2002) Let a, b, c be positive real numbers. Prove that

$$\left(\frac{2a}{b+c}\right)^{\frac{2}{3}} + \left(\frac{2b}{c+a}\right)^{\frac{2}{3}} + \left(\frac{2c}{a+b}\right)^{\frac{2}{3}} \geq 3.$$

Epsilon 73. (APMO 2005) Let a, b, c be positive real numbers with $abc = 8$. Prove that

$$\frac{a^2}{\sqrt{(1+a^3)(1+b^3)}} + \frac{b^2}{\sqrt{(1+b^3)(1+c^3)}} + \frac{c^2}{\sqrt{(1+c^3)(1+a^3)}} \geq \frac{4}{3}$$

Delta 64. [SL 1996 SVN] *Let $a, b,$ and c be positive real numbers such that $abc = 1$. Prove that*

$$\frac{ab}{a^5+b^5+ab} + \frac{bc}{b^5+c^5+bc} + \frac{ca}{c^5+a^5+ca} \leq 1.$$

Delta 65. [SL 1971 YUG] *Prove the inequality*

$$\frac{a_1+a_3}{a_1+a_2} + \frac{a_2+a_4}{a_2+a_3} + \frac{a_3+a_1}{a_3+a_4} + \frac{a_4+a_2}{a_4+a_1} \geq 4$$

where $a_1, a_2, a_3, a_4 > 0$.

There is a simple way to find new bounds for given differentiable functions. We begin to show that every supporting lines are tangent lines in the following sense.

Proposition 5.5. (The Characterization of Supporting Lines) Let f be a real valued function. Let $m, n \in \mathbb{R}$. Suppose that

(1) $f(\alpha) = m\alpha + n$ for some $\alpha \in \mathbb{R}$,
(2) $f(x) \geq mx + n$ for all x in some interval (ϵ_1, ϵ_2) including α, and
(3) f is differentiable at α.

Then, the supporting line $y = mx + n$ of f is the tangent line of f at $x = \alpha$.

Proof. Let us define a function $F : (\epsilon_1, \epsilon_2) \longrightarrow \mathbb{R}$ by $F(x) = f(x) - mx - n$ for all $x \in (\epsilon_1, \epsilon_2)$. Then, F is differentiable at α and we obtain $F'(\alpha) = f'(\alpha) - m$. By the assumption (1) and (2), we see that F has a local minimum at α. So, the first derivative theorem for local extreme values implies that $0 = F'(\alpha) = f'(\alpha) - m$ so that $m = f'(\alpha)$ and that $n = f(\alpha) - m\alpha = f(\alpha) - f'(\alpha)\alpha$. It follows that $y = mx + n = f'(\alpha)(x - \alpha) + f(\alpha)$. $\square$

Proposition 5.6. (Nesbitt) For all positive real numbers a, b, c, we have

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

Proof 9. *We may normalize to $a + b + c = 1$. Note that $0 < a, b, c < 1$. The problem is now to prove*

$$\sum_{\text{cyclic}} f(a) \geq \frac{3}{2}$$

or

$$\frac{f(a) + f(b) + f(c)}{3} \geq f\left(\frac{1}{3}\right),$$

where where $f(x) = \frac{x}{1-x}$. The equation of the tangent line of f at $x = \frac{1}{3}$ is given by $y = \frac{9x-1}{4}$. We claim that the inequality

$$f(x) \geq \frac{9x - 1}{4}$$

holds for all $x \in (0, 1)$. However, it immediately follows from the equality

$$f(x) - \frac{9x - 1}{4} = \frac{(3x - 1)^2}{4(1 - x)}.$$

Now, we conclude that

$$\sum_{\text{cyclic}} \frac{a}{1 - a} \geq \sum_{\text{cyclic}} \frac{9a - 1}{4} = \frac{9}{4} \sum_{\text{cyclic}} a - \frac{3}{4} = \frac{3}{2}.$$

The above argument can be generalized. If a function f has a supporting line at some point on the graph of f, then f satisfies Jensen's Inequality in the following sense.

Theorem 5.2. (Supporting Line Inequality) Let $f : [a, b] \longrightarrow \mathbb{R}$ be a function. Suppose that $\alpha \in [a, b]$ and $m \in \mathbb{R}$ satisfy

$$f(x) \geq m(x - \alpha) + f(\alpha)$$

for all $x \in [a, b]$. Let $\omega_1, \cdots, \omega_n > 0$ with $\omega_1 + \cdots + \omega_n = 1$. Then, the following inequality holds

$$\omega_1 f(x_1) + \cdots + \omega_n f(x_n) \geq f(\alpha)$$

for all $x_1, \cdots, x_n \in [a, b]$ such that $\alpha = \omega_1 x_1 + \cdots + \omega_n x_n$. In particular, we obtain

$$\frac{f(x_1) + \cdots + f(x_n)}{n} \geq f\left(\frac{s}{n}\right),$$

where $x_1, \cdots, x_n \in [a, b]$ with $x_1 + \cdots + x_n = s$ for some $s \in [na, nb]$.

Proof.

$$\begin{aligned} & \omega_1 f(x_1) + \cdots + \omega_n f(x_n) \\ \geq\ & \omega_1[m(x_1 - \alpha) + f(\alpha)] + \cdots + \omega_1[m(x_n - \alpha) + f(\alpha)] \\ =\ & f(\alpha). \end{aligned}$$

□

We can apply the supporting line inequality to deduce Jensen's inequality for differentiable functions.

Lemma 5.1. *Let $f : (a,b) \longrightarrow \mathbb{R}$ be a convex function which is differentiable twice on (a,b). Let $y = l_\alpha(x)$ be the tangent line at $\alpha \in (a,b)$. Then, $f(x) \geq l_\alpha(x)$ for all $x \in (a,b)$. So, the convex function f admits the supporting lines.*

Proof. Let $\alpha \in (a,b)$. We want to show that the tangent line $y = l_\alpha(x) = f'(\alpha)(x - \alpha) + f(\alpha)$ is the supporting line of f at $x = \alpha$ such that $f(x) \geq l_\alpha(x)$ for all $x \in (a,b)$. However, by Taylor's Theorem, we can find a real number θ_x between α and x such that

$$f(x) = f(\alpha) + f'(\alpha)(x-\alpha) + \frac{f''(\theta_x)}{2}(x-\alpha)^2 \geq f(\alpha) + f'(\alpha)(x-\alpha).$$

□

Theorem 5.3. (The Weighted Jensen's Inequality) Let $f : [a,b] \longrightarrow \mathbb{R}$ be a continuous convex function which is differentiable twice on (a,b). Let $\omega_1, \cdots, \omega_n > 0$ with $\omega_1 + \cdots + \omega_n = 1$. For all $x_1, \cdots, x_n \in [a,b]$,

$$\omega_1 f(x_1) + \cdots + \omega_n f(x_n) \geq f(\omega_1\, x_1 + \cdots + \omega_n\, x_n).$$

First Proof. By the continuity of f, we may assume that $x_1, \cdots, x_n \in (a,b)$. Now, let $\mu = \omega_1\, x_1 + \cdots + \omega_n\, x_n$. Then, $\mu \in (a,b)$. By the above lemma, f has the tangent line $y = l_\mu(x) = f'(\mu)(x-\mu) + f(\mu)$ at $x = \mu$ satisfying $f(x) \geq l_\mu(x)$ for all $x \in (a,b)$. Hence, the supporting line inequality shows that

$$\omega_1 f(x_1) + \cdots + \omega_n f(x_n) \geq \omega_1 f(\mu) + \cdots + \omega_n f(\mu) = f(\mu) = f(\omega_1\, x_1 + \cdots + \omega_n\, x_n).$$

□

Non-convex functions can be convex locally and have supporting lines at some points. This means that the supporting line inequality is a powerful tool because we can also produce Jensen-type inequalities for non-convex functions.

Epsilon 74. (Titu Andreescu, Gabriel Dospinescu) Let x, y, and z be real numbers such that $x, y, z \leq 1$ and $x + y + z = 1$. Prove that

$$\frac{1}{1+x^2} + \frac{1}{1+y^2} + \frac{1}{1+z^2} \leq \frac{27}{10}.$$

Epsilon 75. (Japan 1997) Let a, b, and c be positive real numbers. Prove that

$$\frac{(b+c-a)^2}{(b+c)^2+a^2} + \frac{(c+a-b)^2}{(c+a)^2+b^2} + \frac{(a+b-c)^2}{(a+b)^2+c^2} \geq \frac{3}{5}.$$

Any good idea can be stated in fifty words or less. - S. Ulam

6. Homogenizations and Normalizations

Mathematicians do not study objects, but relations between objects.

- H. Poincaré

6.1. Homogenizations. Many inequality problems come with constraints such as $ab = 1$, $xyz = 1$, $x + y + z = 1$. A non-homogeneous symmetric inequality can be transformed into a homogeneous one. Then we apply two powerful theorems: Schur's Inequality and Muirhead's Theorem. We begin with a simple example.

Example 11. (Hungary, 1996) Let a and b be positive real numbers with $a + b = 1$. Prove that

$$\frac{a^2}{a+1} + \frac{b^2}{b+1} \geq \frac{1}{3}.$$

Solution. Using the condition $a+b = 1$, we can reduce the given inequality to homogeneous one:

$$\frac{1}{3} \leq \frac{a^2}{(a+b)(a+(a+b))} + \frac{b^2}{(a+b)(b+(a+b))}$$

or

$$a^2b + ab^2 \leq a^3 + b^3,$$

which follows from

$$(a^3 + b^3) - (a^2b + ab^2) = (a - b)^2(a + b) \geq 0.$$

The equality holds if and only if $a = b = \frac{1}{2}$. □

Theorem 6.1. *Let a_1, a_2, b_1, b_2 be positive real numbers such that $a_1 + a_2 = b_1 + b_2$ and $max(a_1, a_2) \geq max(b_1, b_2)$. Let x and y be nonnegative real numbers. Then, we have*

$$x^{a_1}y^{a_2} + x^{a_2}y^{a_1} \geq x^{b_1}y^{b_2} + x^{b_2}y^{b_1}.$$

Proof. Without loss of generality, we can assume that $a_1 \geq a_2, b_1 \geq b_2, a_1 \geq b_1$. If x or y is zero, then it clearly holds. So, we assume that both x and y are nonzero. It follows from $a_1 + a_2 = b_1 + b_2$ that $a_1 - a_2 = (b_1 - a_2) + (b_2 - a_2)$. It's easy to check

$$\begin{aligned}
& x^{a_1}y^{a_2} + x^{a_2}y^{a_1} - x^{b_1}y^{b_2} - x^{b_2}y^{b_1} \\
= \; & x^{a_2}y^{a_2}\left(x^{a_1-a_2} + y^{a_1-a_2} - x^{b_1-a_2}y^{b_2-a_2} - x^{b_2-a_2}y^{b_1-a_2}\right) \\
= \; & x^{a_2}y^{a_2}\left(x^{b_1-a_2} - y^{b_1-a_2}\right)\left(x^{b_2-a_2} - y^{b_2-a_2}\right) \\
= \; & \frac{1}{x^{a_2}y^{a_2}}\left(x^{b_1} - y^{b_1}\right)\left(x^{b_2} - y^{b_2}\right) \geq 0.
\end{aligned}$$

□

Remark 6.1. *When does the equality hold in the above theorem?*

We now introduce two summation notations. Let $\mathcal{P}(x, y, z)$ be a three variables function of x, y, z. Let us define

$$\sum_{\text{cyclic}} \mathcal{P}(x, y, z) = \mathcal{P}(x, y, z) + \mathcal{P}(y, z, x) + \mathcal{P}(z, x, y)$$

and

$$\sum_{\text{sym}} \mathcal{P}(x, y, z) = \mathcal{P}(x, y, z) + \mathcal{P}(x, z, y) + \mathcal{P}(y, x, z) + \mathcal{P}(y, z, x) + \mathcal{P}(z, x, y) + \mathcal{P}(z, y, x).$$

Here, we have some examples:

$$\sum_{\text{cyclic}} x^3y = x^3y + y^3z + z^3x, \quad \sum_{\text{sym}} x^3 = 2(x^3 + y^3 + z^3),$$

$$\sum_{\text{sym}} x^2y = x^2y + x^2z + y^2z + y^2x + z^2x + z^2y, \quad \sum_{\text{sym}} xyz = 6xyz$$

Example 12. *Let x, y, z be positive real numbers. Show the cyclic inequality*

$$\frac{x^2}{y^2} + \frac{y^2}{z^2} + \frac{z^2}{x^2} \geq \frac{x}{y} + \frac{y}{z} + \frac{z}{x}.$$

Third Solution. We break the homogeneity. After the substitution $a = \frac{x}{y}, b = \frac{y}{z}, c = \frac{z}{x}$, it becomes

$$a^2 + b^2 + c^2 \geq a + b + c.$$

Using the constraint $abc = 1$, we now impose the homogeneity to this as follows:

$$a^2 + b^2 + c^2 \geq (abc)^{\frac{1}{3}} (a + b + c).$$

After setting $a = x^3, b = y^3, c = z^3$ with $x, y, z > 0$, it then becomes

$$x^6 + y^6 + z^6 \geq x^4yz + xy^4z + xyz^4.$$

We now deduce

$$\sum_{\text{cyclic}} x^6 = \sum_{\text{cyclic}} \frac{x^6 + y^6}{2} \geq \sum_{\text{cyclic}} \frac{x^4y^2 + x^2y^4}{2} = \sum_{\text{cyclic}} x^4 \left(\frac{y^2 + z^2}{2} \right) \geq \sum_{\text{cyclic}} x^4yz.$$

$\square$

Epsilon 76. [IMO 1984/1 FRG] *Let x, y, z be nonnegative real numbers such that $x+y+z = 1$. Prove that*

$$0 \leq xy + yz + zx - 2xyz \leq \frac{7}{27}.$$

Epsilon 77. [LL 1992 UNK] (Iran 1998) Prove that, for all $x, y, z > 1$ such that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2$,

$$\sqrt{x + y + z} \geq \sqrt{x - 1} + \sqrt{y - 1} + \sqrt{z - 1}.$$

6.2. **Schur and Muirhead.**

Theorem 6.2. (Schur's Inequality) Let x, y, z be nonnegative real numbers. For any $r > 0$, we have

$$\sum_{\text{cyclic}} x^r(x-y)(x-z) \geq 0.$$

Proof. Since the inequality is symmetric in the three variables, we may assume without loss of generality that $x \geq y \geq z$. Then the given inequality may be rewritten as

$$(x-y)[x^r(x-z) - y^r(y-z)] + z^r(x-z)(y-z) \geq 0,$$

and every term on the left-hand side is clearly nonnegative. □

Remark 6.2. *When does the equality hold in Schur's Inequality?*

Delta 66. *Disprove the following proposition: for all $a, b, c, d \geq 0$ and $r > 0$, we have*

$$a^r(a-b)(a-c)(a-d)+b^r(b-c)(b-d)(b-a)+c^r(c-a)(c-c)(a-d)+d^r(d-a)(d-b)(d-c) \geq 0.$$

Delta 67. [LL 1971 HUN] *Let a, b, c, d, e be real numbers. Prove the expression*

$$\begin{aligned} & (a-b)(a-c)(a-d)(a-e)+(b-a)(b-c)(b-d)(b-e) \\ + \quad & (c-a)(c-b)(c-d)(c-e)+(d-a)(d-b)(d-c)(a-e) \\ + \quad & (e-a)(e-b)(e-c)(e-d) \end{aligned}$$

is nonnegative.

The following special case of Schur's Inequality is useful:

$$\sum_{\text{cyclic}} x(x-y)(x-z) \geq 0 \Leftrightarrow 3xyz + \sum_{\text{cyclic}} x^3 \geq \sum_{\text{sym}} x^2y \Leftrightarrow \sum_{\text{sym}} xyz + \sum_{\text{sym}} x^3 \geq 2\sum_{\text{sym}} x^2y.$$

Epsilon 78. *Let x, y, z be nonnegative real numbers. Then, we have*

$$3xyz + x^3 + y^3 + z^3 \geq 2\left((xy)^{\frac{3}{2}} + (yz)^{\frac{3}{2}} + (zx)^{\frac{3}{2}}\right).$$

Epsilon 79. *Let $t \in (0, 3]$. For all $a, b, c \geq 0$, we have*

$$(3-t) + t(abc)^{\frac{2}{t}} + \sum_{\text{cyclic}} a^2 \geq 2\sum_{\text{cyclic}} ab.$$

Epsilon 80. (APMO 2004/5) Prove that, for all positive real numbers a, b, c,

$$(a^2+2)(b^2+2)(c^2+2) \geq 9(ab+bc+ca).$$

Epsilon 81. [IMO 2000/2 USA] *Let a, b, c be positive numbers such that $abc = 1$. Prove that*

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1.$$

Epsilon 82. (Tournament of Towns 1997) Let a, b, c be positive numbers such that $abc = 1$. Prove that

$$\frac{1}{a+b+1} + \frac{1}{b+c+1} + \frac{1}{c+a+1} \leq 1.$$

Delta 68. [TZ, p.142] Prove that for any acute triangle ABC,

$$\cot^3 A + \cot^3 B + \cot^3 C + 6\cot A\cot B\cot C \geq \cot A + \cot B + \cot C.$$

Delta 69. [IN, p.103] Let a, b, c be the lengths of a triangle. Prove that

$$a^2b + a^2c + b^2c + b^2a + c^2a + c^2b > a^3 + b^3 + c^3 + 2abc.$$

Delta 70. (Surányi's Inequality) Show that, for all $x_1, \cdots, x_n \geq 0$,

$$(n-1)\left({x_1}^n + \cdots {x_n}^n\right) + nx_1\cdots x_n \geq (x_1 + \cdots + x_n)\left({x_1}^{n-1} + \cdots {x_n}^{n-1}\right).$$

Epsilon 83. (Muirhead's Theorem) Let $a_1, a_2, a_3, b_1, b_2, b_3$ be non-negative real numbers such that

$$a_1 \geq a_2 \geq a_3,\ b_1 \geq b_2 \geq b_3,\ a_1 \geq b_1,\ a_1 + a_2 \geq b_1 + b_2,\ a_1 + a_2 + a_3 = b_1 + b_2 + b_3.$$

(In this case, we say that the vector $\mathrm{a} = (a_1, a_2, a_3)$ majorizes the vector $\mathrm{b} = (b_1, b_2, b_3)$ and write $\mathrm{a} \succ \mathrm{b}$.) For all positive real numbers x, y, z, we have

$$\sum_{\text{sym}} x^{a_1} y^{a_2} z^{a_3} \geq \sum_{\text{sym}} x^{b_1} y^{b_2} z^{b_3}.$$

Remark 6.3. *The equality holds if and only if $x = y = z$. However, if we allow $x = 0$ or $y = 0$ or $z = 0$, then one may easily check that the equality holds (after assuming $a_1, a_2, a_3 > 0$ and $b_1, b_2, b_3 > 0$) if and only if*

$$x = y = z \text{ or } x = y,\ z = 0 \text{ or } y = z,\ x = 0 \text{ or } z = x,\ y = 0.$$

We can apply Muirhead's Theorem to establish Nesbitt's Inequality.

Proposition 6.1. (Nesbitt) For all positive real numbers a, b, c, we have

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

Proof 10. *Clearing the denominators of the inequality, it becomes*

$$2 \sum_{\text{cyclic}} a(a+b)(a+c) \geq 3(a+b)(b+c)(c+a)$$

or

$$\sum_{\text{sym}} a^3 \geq \sum_{\text{sym}} a^2 b.$$

Epsilon 84. **[IMO 1995/2 RUS]** *Let a, b, c be positive numbers such that $abc = 1$. Prove that*

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

Epsilon 85. (Iran 1996) Let x, y, z be positive real numbers. Prove that

$$(xy + yz + zx)\left(\frac{1}{(x+y)^2} + \frac{1}{(y+z)^2} + \frac{1}{(z+x)^2}\right) \geq \frac{9}{4}.$$

Epsilon 86. *Let x, y, z be nonnegative real numbers with $xy + yz + zx = 1$. Prove that*

$$\frac{1}{x+y} + \frac{1}{y+z} + \frac{1}{z+x} \geq \frac{5}{2}.$$

Epsilon 87. [SC] If m_a,m_b,m_c are medians and r_a,r_b,r_c the exradii of a triangle, prove that

$$\frac{r_a r_b}{m_a m_b} + \frac{r_b r_c}{m_b m_c} + \frac{r_c r_a}{m_c m_a} \geq 3.$$

We now offer a criterion for the homogeneous symmetric polynomial inequalities with degree 3. It is a direct consequence of Schur's Inequality and Muirhead's Theorem.

Epsilon 88. *Let $\mathcal{P}(u, v, w) \in \mathbb{R}[u, v, w]$ be a homogeneous symmetric polynomial with degree 3. Then the following two statements are equivalent.*

(a) $\mathcal{P}(1, 1, 1),\ \mathcal{P}(1, 1, 0),\ \mathcal{P}(1, 0, 0) \geq 0$.
(b) $\mathcal{P}(x, y, z) \geq 0$ *for all* $x, y, z \geq 0$.

Example 13. **[IMO 1984/1 FRG]** *Let x, y, z be nonnegative real numbers such that $x + y + z = 1$. Prove that*

$$0 \leq xy + yz + zx - 2xyz \leq \frac{7}{27}.$$

Solution. Using $x + y + z = 1$, we convert the given inequality to the equivalent form:

$$0 \leq (xy + yz + zx)(x + y + z) - 2xyz \leq \frac{7}{27}(x + y + z)^3.$$

Let us define $\mathcal{L}(u, v, w), \mathcal{R}(u, v, w) \in \mathbb{R}[u, v, w]$ by

$$\mathcal{L}(u, v, w) = (uv + vw + wu)(u + v + w) - 2uvw,$$

$$\mathcal{R}(u, v, w) = \frac{7}{27}(u + v + w)^3 - (uv + vw + wu)(u + v + w) + 2uvw.$$

However, one may easily check that

$$\mathcal{L}(1, 1, 1) = 7,\ \mathcal{L}(1, 1, 0) = 2,\ \mathcal{L}(1, 0, 0) = 0,$$

$$\mathcal{R}(1, 1, 1) = 0,\ \mathcal{R}(1, 1, 0) = \frac{2}{27},\ \mathcal{R}(1, 0, 0) = \frac{7}{27}.$$

□

In other words, we don't need to employ Schur's Inequality and Muirhead's Theorem to get a straightforward result.

Delta 71. (M. S. Klamkin) Determine the maximum and minimum values of

$$x^2 + y^2 + z^2 + \lambda xyz$$

where $x + y + z = 1$, $x, y, z \geq 0$, and λ is a given constant.

Delta 72. (W. Janous) Let $x, y, z \geq 0$ with $x + y + z = 1$. For fixed real numbers $a \geq 0$ and b, determine the maximum $c = c(a, b)$ such that

$$a + bxyz \geq c(xy + yz + zx).$$

As a corollary of the above criterion, we obtain the following proposition for homogeneous symmetric polynomial inequalities for the triangles :

Theorem 6.3. (K. B. Stolarsky) Let $\mathcal{P}(u, v, w)$ be a real symmetric form of degree 3. If we have

$$\mathcal{P}(1, 1, 1),\ \mathcal{P}(1, 1, 0),\ \mathcal{P}(2, 1, 1) \geq 0,$$

then $\mathcal{P}(a, b, c) \geq 0$, where a, b, c are the lengths of the sides of a triangle.

Proof. Employ The Ravi Substitution together with the above crieterion. We leave the details for the readers. For an alternative proof, see [KS]. □

Delta 73. (China 2007) Let a, b, c be the lengths of a triangle with $a+b+c = 3$. Determine the minimum value of

$$a^2 + b^2 + c^2 + \frac{4abc}{3}.$$

As noted in [KS], applying Stolarsky's Crieterion, we obtain various cubic inequalities in triangle geometry.

Example 14. *Let a, b, c be the lengths of the sides of a triangle. Let s be the semiperimeter of the triangle. Then, the following inequalities holds.*

(a) $4(ab + bc + ca) > (a + b + c)^2 \geq 3(ab + bc + ca)$
(b) [DM] $a^2 + b^2 + c^2 \geq \frac{36}{35}\left(s^2 + \frac{abc}{s}\right)$
(c) [AP] $abc \geq 8(s - a)(s - b)(s - c)$
(d) [EC] $8abc \geq (a + b)(b + c)(c + a)$
(e) [AP] $8(a^3 + b^3 + c^3) \geq 3(a + b)(b + c)(c + a)$
(f) [MC] $2(a + b + c)(a^2 + b^2 + c^2) \geq 3(a^3 + b^3 + c^3 + 3abc)$
(g) $\frac{3}{2}abc \geq a^2(s - a) + b^2(s - b) + c^2(s - c) > abc$
(h) $bc(b + c) + ca(c + a) + ab(a + b) \geq 48(s - a)(s - b)(s - c)$
(i) $\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} \geq \frac{9}{s}$
(j) [AN, MP] $2 > \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$

(k) $\frac{9}{2} > \frac{s+a}{b+c} + \frac{s+b}{c+a} + \frac{s+c}{a+b} \geq \frac{15}{4}$
(l) [SR1] $5[ab(a+b) + bc(b+c) + ca(c+a)] - 3abc \geq (a+b+c)^3$

Proof. We only check the left hand side inequality in (j). One may easily check that it is equivalent to the cubic inequality $\mathcal{T}(a,b,c) \geq 0$, where

$$\mathcal{T}(a,b,c) = 2(a+b)(b+c)(c+a) - (a+b)(b+c)(c+a)\left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right).$$

Since $\mathcal{T}(1,1,1) = 4$, $\mathcal{T}(1,1,0) = 0$, and $\mathcal{T}(2,1,1) = 6$, the result follows from Stolarsky's Criterion. For alternative proofs, see [BDJMV]. $\square$

6.3. **Normalizations.** In the previous subsections, we transformed non-homogeneous inequalities into homogeneous ones. On the other hand, homogeneous inequalities also can be normalized in various ways. We offer two alternative solutions of the problem 8 by normalizations :

Epsilon 89. [IMO 2001/2 KOR] *Let a, b, c be positive real numbers. Prove that*

$$\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ca}}+\frac{c}{\sqrt{c^2+8ab}}\geq 1.$$

Epsilon 90. [IMO 1983/6 USA] *Let a, b, c be the lengths of the sides of a triangle. Prove that*

$$a^2b(a-b)+b^2c(b-c)+c^2a(c-a)\geq 0.$$

Epsilon 91. (KMO Winter Program Test 2001) Prove that, for all $a, b, c > 0$,

$$\sqrt{(a^2b+b^2c+c^2a)\,(ab^2+bc^2+ca^2)}\geq abc+\sqrt[3]{(a^3+abc)\,(b^3+abc)\,(c^3+abc)}$$

Epsilon 92. [IMO 1999/2 POL] *Let n be an integer with $n \geq 2$. (a) Determine the least constant C such that the inequality*

$$\sum_{1\leq i<j\leq n} x_ix_j(x_i^2+x_j^2)\leq C\left(\sum_{1\leq i\leq n} x_i\right)^4$$

holds for all real numbers $x_1, \cdots, x_n \geq 0$.
(b) For this constant C, determine when equality holds.

Delta 74. [SL 1991 POL] *Let n be a given integer with $n \geq 2$. Find the maximum value of*

$$\sum_{1\leq i<j\leq n} x_ix_j(x_i+x_j),$$

where $x_1, \cdots, x_n \geq 0$ and $x_1 + \cdots + x_n = 1$.

We close this subsection with another proofs of Nesbitt's Inequality.

Proposition 6.2. (Nesbitt) For all positive real numbers a, b, c, we have

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}.$$

Proof 11. *We may normalize to $a + b + c = 1$. Note that $0 < a, b, c < 1$. The problem is now to prove*

$$\sum_{\text{cyclic}}\frac{a}{b+c}=\sum_{\text{cyclic}} f(a)\geq\frac{3}{2},\ \ where\ \ f(x)=\frac{x}{1-x}.$$

Since f is convex on $(0, 1)$, Jensen's Inequality shows that

$$\frac{1}{3}\sum_{\text{cyclic}} f(a)\geq f\left(\frac{a+b+c}{3}\right)=f\left(\frac{1}{3}\right)=\frac{1}{2}\quad or\quad \sum_{\text{cyclic}} f(a)\geq\frac{3}{2}.$$

Proof 12. (Cao Minh Quang) Assume that $a + b + c = 1$. Note that $ab + bc + ca \leq \frac{1}{3}(a+b+c)^2 = \frac{1}{3}$. More strongly, we establish that

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq 3-\frac{9}{2}(ab+bc+ca)$$

or

$$\left(\frac{a}{b+c}+\frac{9a(b+c)}{4}\right)+\left(\frac{b}{c+a}+\frac{9b(c+a)}{4}\right)+\left(\frac{c}{a+b}+\frac{9c(a+b)}{4}\right)\geq 3.$$

The AM-GM inequality shows that

$$\sum_{\text{cyclic}}\frac{a}{b+c}+\frac{9a(b+c)}{4}\geq\sum_{\text{cyclic}} 2\sqrt{\frac{a}{b+c}\cdot\frac{9a(b+c)}{4}}=\sum_{\text{cyclic}} 3a=3.$$

6.4. **Cauchy-Schwarz and Hölder.** We begin with the following famous theorem:

Theorem 6.4. (The Cauchy-Schwarz Inequality) Whenever $a_1, \cdots, a_n, b_1, \cdots, b_n \in \mathbb{R}$, we have

$$({a_1}^2 + \cdots + {a_n}^2)({b_1}^2 + \cdots + {b_n}^2) \geq (a_1 b_1 + \cdots + a_n b_n)^2.$$

First Proof. Let $A = \sqrt{{a_1}^2 + \cdots + {a_n}^2}$ and $B = \sqrt{{b_1}^2 + \cdots + {b_n}^2}$. In the case when $A = 0$, we get $a_1 = \cdots = a_n = 0$. Thus, the given inequality clearly holds. From now on, we assume that $A, B > 0$. Since the inequality is homogeneous, we may normalize to

$$1 = {a_1}^2 + \cdots + {a_n}^2 = {b_1}^2 + \cdots + {b_n}^2.$$

We now need to to show that

$$|a_1 b_1 + \cdots + a_n b_n| \leq 1.$$

Indeed, we deduce

$$|a_1 b_1 + \cdots + a_n b_n| \leq |a_1 b_1| + \cdots + |a_n b_n| \leq \frac{{a_1}^2 + {b_1}^2}{2} + \cdots + \frac{{a_n}^2 + {b_n}^2}{2} = 1.$$

□

Second Proof. It immediately follows from The Lagrange Identity:

$$\left(\sum_{i=1}^{n} {a_i}^2\right)\left(\sum_{i=1}^{n} {b_i}^2\right) - \left(\sum_{i=1}^{n} a_i b_i\right)^2 = \sum_{1 \leq i < j \leq n} (a_i b_j - a_j b_i)^2.$$

□

Delta 75. [IMO 2003/5 IRL] *Let n be a positive integer and let $x_1 \leq \cdots \leq x_n$ be real numbers. Prove that*

$$\left(\sum_{1 \leq i,j \leq n} |x_i - x_j|\right)^2 \leq \frac{2\left(n^2 - 1\right)}{3} \sum_{1 \leq i,j \leq n} (x_i - x_j)^2.$$

Show that the equality holds if and only if $x_1, \cdots, x_n$ is an arithmetic progression.

Delta 76. (Darij Grinberg) Suppose that $0 < a_1 \leq \cdots \leq a_n$ and $0 < b_1 \leq \cdots \leq b_n$ be real numbers. Show that

$$\frac{1}{4}\left(\sum_{k=1}^{n} a_k\right)^2 \left(\sum_{k=1}^{n} b_k\right)^2 > \left(\sum_{k=1}^{n} {a_k}^2\right)\left(\sum_{k=1}^{n} {b_k}^2\right) - \left(\sum_{k=1}^{n} a_k b_k\right)^2$$

Delta 77. [LL 1971 AUT] *Let a, b, c be positive real numbers, $0 < a \leq b \leq c$. Prove that for any $x, y, z > 0$ the following inequality holds:*

$$\frac{(a+c)^2}{4ac}(x+y+z)^2 \geq (ax + by + cz)\left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right).$$

Delta 78. [LL 1987 AUS] *Let $a_1, a_2, a_3, b_1, b_2, b_3$ be positive real numbers. Prove that*

$$(a_1 b_2 + a_1 b_3 + a_2 b_1 + a_2 b_3 + a_3 b_1 + a_3 b_2)^2 \geq 4\left(a_1 a_2 + a_2 a_3 + a_3 a_1\right)\left(b_1 b_2 + b_2 b_3 + b_3 b_1\right)$$

and show that the two sides of the inequality are equal if and only if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$.

Delta 79. [PF] Let $a_1, \cdots, a_n, b_1, \cdots, b_n \in \mathbb{R}$. Suppose that $x \in [0, 1]$. Show that

$$\left(\sum_{i=1}^{n} {a_i}^2 + 2x \sum_{i<j} a_i a_j\right)\left(\sum_{i=1}^{n} {b_i}^2 + 2x \sum_{i<j} b_i b_j\right) \geq \left(\sum_{i=1}^{n} a_i b_i + x \sum_{i \leq j} a_i b_j\right)^2.$$

Delta 80. *Let $a_1, \cdots, a_n, b_1, \cdots, b_n$ be positive real numbers. Show that*

$$\begin{cases} (1)\ \sqrt{(a_1+\cdots+a_n)(b_1+\cdots+b_n)} \geq \sqrt{a_1b_1}+\cdots+\sqrt{a_nb_n}, \\ (2)\ \frac{{a_1}^2}{b_1}+\cdots+\frac{{a_n}^2}{b_n} \geq \frac{(a_1+\cdots+a_n)^2}{b_1+\cdots+b_n}, \\ (3)\ \frac{a_1}{{b_1}^2}+\cdots+\frac{a_n}{{b_n}^2} \geq \frac{1}{a_1+\cdots+a_n}\left(\frac{a_1}{b_1}+\cdots+\frac{a_n}{b_n}\right)^2, \\ (4)\ \frac{a_1}{b_1}+\cdots+\frac{a_n}{b_n} \geq \frac{(a_1+\cdots+a_n)^2}{a_1b_1+\cdots+a_nb_n}. \end{cases}$$

Delta 81. [SL 1993 USA] *Prove that*

$$\frac{a}{b+2c+3d}+\frac{b}{c+2d+3a}+\frac{c}{d+2a+3b}+\frac{d}{a+2b+3c} \geq \frac{2}{3}$$

for all positive real numbers a, b, c, d.

Epsilon 93. (APMO 1991) Let $a_1, \cdots, a_n, b_1, \cdots, b_n$ be positive real numbers such that $a_1+\cdots+a_n = b_1+\cdots+b_n$. Show that

$$\frac{{a_1}^2}{a_1+b_1}+\cdots+\frac{{a_n}^2}{a_n+b_n} \geq \frac{a_1+\cdots+a_n}{2}.$$

Epsilon 94. *Let $a, b \geq 0$ with $a+b=1$. Prove that*

$$\sqrt{a^2+b}+\sqrt{a+b^2}+\sqrt{1+ab} \leq 3.$$

Show that the equality holds if and only if $(a,b)=(1,0)$ or $(a,b)=(0,1)$.

Epsilon 95. [LL 1992 UNK] (Iran 1998) Prove that, for all $x, y, z > 1$ such that $\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = 2$,

$$\sqrt{x+y+z} \geq \sqrt{x-1}+\sqrt{y-1}+\sqrt{z-1}.$$

We now apply The Cauchy-Schwarz Inequality to prove Nesbitt's Inequality.

Proposition 6.3. (Nesbitt) For all positive real numbers a, b, c, we have

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq \frac{3}{2}.$$

Proof 13. *Applying The Cauchy-Schwarz Inequality, we have*

$$((b+c)+(c+a)+(a+b))\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right) \geq 3^2.$$

It follows that

$$\frac{a+b+c}{b+c}+\frac{a+b+c}{c+a}+\frac{a+b+c}{a+b} \geq \frac{9}{2}$$

or

$$3+\sum_{\text{cyclic}} \frac{a}{b+c} \geq \frac{9}{2}.$$

Proof 14. *The Cauchy-Schwarz Inequality yields*

$$\sum_{\text{cyclic}} \frac{a}{b+c} \sum_{\text{cyclic}} a(b+c) \geq \left(\sum_{\text{cyclic}} a\right)^2$$

or

$$\sum_{\text{cyclic}} \frac{a}{b+c} \geq \frac{(a+b+c)^2}{2(ab+bc+ca)} \geq \frac{3}{2}.$$

Epsilon 96. (Gazeta Matematicã) Prove that, for all $a, b, c > 0$,

$$\sqrt{a^4+a^2b^2+b^4}+\sqrt{b^4+b^2c^2+c^4}+\sqrt{c^4+c^2a^2+a^4} \geq a\sqrt{2a^2+bc}+b\sqrt{2b^2+ca}+c\sqrt{2c^2+ab}.$$

Epsilon 97. (KMO Winter Program Test 2001) Prove that, for all $a, b, c > 0$,

$$\sqrt{(a^2b+b^2c+c^2a)(ab^2+bc^2+ca^2)} \geq abc+\sqrt[3]{(a^3+abc)(b^3+abc)(c^3+abc)}$$

Epsilon 98. (Andrei Ciupan) Let a, b, c be positive real numbers such that

$$\frac{1}{a+b+1}+\frac{1}{b+c+1}+\frac{1}{c+a+1}\geq 1.$$

Show that $a+b+c\geq ab+bc+ca$.

We now illustrate normalization techniques to establish classical theorems. Using the same idea in the proof of The Cauchy-Schwarz Inequality, we find a natural generalization :

Theorem 6.5. *Let $a_{ij}(i,j=1,\cdots,n)$ be positive real numbers. Then, we have*

$$({a_{11}}^n+\cdots+{a_{1n}}^n)\cdots({a_{n1}}^n+\cdots+{a_{nn}}^n)\geq(a_{11}a_{21}\cdots a_{n1}+\cdots+a_{1n}a_{2n}\cdots a_{nn})^n.$$

Proof. The inequality is homogeneous. We make the normalizations:

$$({a_{i1}}^n+\cdots+{a_{in}}^n)^{\frac{1}{n}}=1$$

or

$${a_{i1}}^n+\cdots+{a_{in}}^n=1,$$

for all $i=1,\cdots,n$. Then, the inequality takes the form

$$a_{11}a_{21}\cdots a_{n1}+\cdots+a_{1n}a_{2n}\cdots a_{nn}\leq 1$$

or

$$\sum_{i=1}^{n}a_{i1}\cdots a_{in}\leq 1.$$

Hence, it suffices to show that, for all $i=1,\cdots,n$,

$$a_{i1}\cdots a_{in}\leq\frac{1}{n}$$

where ${a_{i1}}^n+\cdots+{a_{in}}^n=1$. To finish the proof, it remains to show the following homogeneous inequality. □

Theorem 6.6. (The AM-GM Inequality) Let $a_1,\cdots,a_n$ be positive real numbers. Then, we have

$$\frac{a_1+\cdots+a_n}{n}\geq\sqrt[n]{a_1\cdots a_n}.$$

Proof. Since it's homogeneous, we may rescale $a_1,\cdots,a_n$ so that $a_1\cdots a_n=1$. [14] We want to show that

$$a_1\cdots a_n=1\implies a_1+\cdots+a_n\geq n.$$

The proof is by induction on n. If $n=1$, it's trivial. If $n=2$, then we get $a_1+a_2-2=a_1+a_2-2\sqrt{a_1a_2}=(\sqrt{a_1}-\sqrt{a_2})^2\geq 0$. Now, we assume that it holds for some positive integer $n\geq 2$. And let $a_1,\cdots,a_{n+1}$ be positive numbers such that $a_1\cdots a_na_{n+1}$=1. We may assume that $a_1\geq 1\geq a_2$. (Why?) It follows that $a_1a_2+1-a_1-a_2=(a_1-1)(a_2-1)\leq 0$ so that $a_1a_2+1\leq a_1+a_2$. Since $(a_1a_2)a_3\cdots a_n=1$, by the induction hypothesis, we have

$$a_1a_2+a_3+\cdots+a_{n+1}\geq n.$$

It follows that $a_1+a_2-1+a_3+\cdots+a_{n+1}\geq n$. □

[14] Set $x_i=\frac{a_i}{(a_1\cdots a_n)^{\frac{1}{n}}}$ $(i=1,\cdots,n)$. Then, we get $x_1\cdots x_n=1$ and it becomes $x_1+\cdots+x_n\geq n$.

We now make simple observation. Let $a, b > 0$ and $m, n \in \mathbf{N}$. Take $x_1 = \cdots = x_m = a$ and $x_{m+1} = \cdots = x_{x_{m+n}} = b$. Applying the AM-GM inequality to $x_1, \cdots, x_{m+n} > 0$, we obtain

$$\frac{ma + nb}{m+n} \geq (a^m b^n)^{\frac{1}{m+n}} \quad \text{or} \quad \frac{m}{m+n} a + \frac{n}{m+n} b \geq a^{\frac{m}{m+n}} b^{\frac{n}{m+n}}.$$

Hence, for all positive rational numbers ω_1 and ω_2 with $\omega_1 + \omega_2 = 1$, we get

$$\omega_1\, a + \omega_2\, b \geq a^{\,\omega_1} b^{\,\omega_2}.$$

We now immediately have

Theorem 6.7. *Let ω_1, $\omega_2 > 0$ with $\omega_1 + \omega_2 = 1$. For all x, $y > 0$, we have*

$$\omega_1\, x + \omega_2\, y \geq x^{\,\omega_1} y^{\,\omega_2}.$$

Proof. We can choose a sequence $a_1, a_2, a_3, \cdots \in (0, 1)$ of rational numbers such that

$$\lim_{n\to\infty} a_n = \omega_1.$$

Set $b_i = 1 - a_i$, where $i \in \mathbb{N}$. Then, $b_1, b_2, b_3, \cdots \in (0, 1)$ is a sequence of rational numbers with

$$\lim_{n\to\infty} b_n = \omega_2.$$

From the previous observation, we have $a_n\ x + b_n\ y \geq x^{a_n} y^{b_n}$. By taking the limits to both sides, we get the result. $\square$

We can extend the above arguments to the n-variables.

Theorem 6.8. (The Weighted AM-GM Inequality) Let $\omega_1, \cdots, \omega_n > 0$ with $\omega_1 + \cdots + \omega_n = 1$. For all $x_1, \cdots, x_n > 0$, we have

$$\omega_1\, x_1 + \cdots + \omega_n\, x_n \geq x_1^{\;\omega_1} \cdots x_n^{\;\omega_n}.$$

Since we now get the weighted version of The AM-GM Inequality, we establish weighted version of The Cauchy-Schwarz Inequality.

Epsilon 99. (Hölder's Inequality) Let x_{ij} $(i = 1, \cdots, m, j = 1, \cdots n)$ be positive real numbers. Suppose that $\omega_1, \cdots, \omega_n$ are positive real numbers satisfying $\omega_1 + \cdots + \omega_n = 1$. Then, we have

$$\prod_{j=1}^{n} \left(\sum_{i=1}^{m} x_{ij} \right)^{\omega_j} \geq \sum_{i=1}^{m} \left(\prod_{j=1}^{n} x_{ij}^{\;\omega_j} \right).$$

7. Convexity and Its Applications

The art of doing mathematics consists in finding that special case which contains all the germs of generality.

- D. Hilbert

7.1. **Jensen's Inequality.** In the previous section, we deduced the weighted AM-GM inequality from The AM-GM Inequality. We use the same idea to study the following functional inequalities.

Epsilon 100. *Let $f : [a,b] \longrightarrow \mathbb{R}$ be a continuous function. Then, the followings are equivalent.*

(1) For all $n \in \mathbb{N}$, the following inequality holds.

$$\omega_1 f(x_1) + \cdots + \omega_n f(x_n) \geq f(\omega_1\, x_1 + \cdots + \omega_n\, x_n)$$

for all $x_1, \cdots, x_n \in [a,b]$ and $\omega_1, \cdots, \omega_n > 0$ with $\omega_1 + \cdots + \omega_n = 1$.

(2) For all $n \in \mathbb{N}$, the following inequality holds.

$$r_1 f(x_1) + \cdots + r_n f(x_n) \geq f(r_1\, x_1 + \cdots + r_n\, x_n)$$

for all $x_1, \cdots, x_n \in [a,b]$ and $r_1, \cdots, r_n \in \mathbb{Q}^+$ with $r_1 + \cdots + r_n = 1$.

(3) For all $N \in \mathbb{N}$, the following inequality holds.

$$\frac{f(y_1) + \cdots + f(y_N)}{N} \geq f\left(\frac{y_1 + \cdots + y_N}{N}\right)$$

for all $y_1, \cdots, y_N \in [a,b]$.

(4) For all $k \in \{0,1,2,\cdots\}$, the following inequality holds.

$$\frac{f(y_1) + \cdots + f(y_{2^k})}{2^k} \geq f\left(\frac{y_1 + \cdots + y_{2^k}}{2^k}\right)$$

for all $y_1, \cdots, y_{2^k} \in [a,b]$.

(5) We have $\frac{1}{2}f(x) + \frac{1}{2}f(y) \geq f\left(\frac{x+y}{2}\right)$ for all $x, y \in [a,b]$.

(6) We have $\lambda f(x) + (1-\lambda) f(y) \geq f(\lambda x + (1-\lambda) y)$ for all $x, y \in [a,b]$ and $\lambda \in (0,1)$.

Definition 7.1. *A real valued function $f : [a,b] \longrightarrow \mathbb{R}$ is said to be convex if the inequality*

$$\lambda f(x) + (1-\lambda) f(y) \geq f(\lambda x + (1-\lambda) y)$$

holds for all $x, y \in [a,b]$ and $\lambda \in (0,1)$.

The above proposition says that

Corollary 7.1. (Jensen's Inequality) If $f : [a,b] \longrightarrow \mathbb{R}$ is a continuous convex function, then for all $x_1, \cdots, x_n \in [a,b]$, we have

$$\frac{f(x_1) + \cdots + f(x_n)}{n} \geq f\left(\frac{x_1 + \cdots + x_n}{n}\right).$$

Delta 82. [SL 1998 AUS] *Let $r_1, \cdots, r_n$ be real numbers greater than or equal to 1. Prove that*

$$\frac{1}{r_1+1} + \cdots + \frac{1}{r_n+1} \geq \frac{n}{\sqrt[n]{r_1 \cdots r_n} + 1}.$$

Corollary 7.2. (The Weighted Jensen's Inequality) Let $f : [a,b] \longrightarrow \mathbb{R}$ be a continuous convex function. Let $\omega_1, \cdots, \omega_n > 0$ with $\omega_1 + \cdots + \omega_n = 1$. For all $x_1, \cdots, x_n \in [a,b]$, we have

$$\omega_1 f(x_1) + \cdots + \omega_n f(x_n) \geq f(\omega_1\, x_1 + \cdots + \omega_n\, x_n).$$

In fact, we can almost drop the continuity of f. As an exercise, show that every convex function on $[a, b]$ is continuous on (a, b). Hence, every convex function on $\mathbb{R}$ is continuous on $\mathbb{R}$.

Corollary 7.3. (The Convexity Criterion I) If a continuous function $f : [a, b] \longrightarrow \mathbb{R}$ satisfies the midpoint convexity
$$\frac{f(x) + f(y)}{2} \geq f\left(\frac{x+y}{2}\right)$$
for all $x, y \in [a, b]$, then the function f is convex on $[a, b]$.

Delta 83. (The Convexity Criterion II) Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function which are differentiable twice in (a, b). Show that (1) $f''(x) \geq 0$ for all $x \in (a, b)$ if and only if (2) f is convex on (a, b).

We now present an inductive proof of The Weighted Jensen's Inequality. It turns out that we can completely drop the continuity of f.

Third Proof. It clearly holds for $n = 1, 2$. We now assume that it holds for some $n \in \mathbb{N}$. Let $x_1, \cdots, x_n, x_{n+1} \in [a, b]$ and $\omega_1, \cdots, \omega_{n+1} > 0$ with $\omega_1 + \cdots + \omega_{n+1} = 1$. Since we have the equality
$$\frac{\omega_1}{1 - \omega_{n+1}} + \cdots + \frac{\omega_n}{1 - \omega_{n+1}} = 1,$$
by the induction hypothesis, we obtain
$$\begin{aligned}
& \omega_1 f(x_1) + \cdots + \omega_{n+1} f(x_{n+1}) \\
= \; & (1 - \omega_{n+1}) \left(\frac{\omega_1}{1 - \omega_{n+1}} f(x_1) + \cdots + \frac{\omega_n}{1 - \omega_{n+1}} f(x_n) \right) + \omega_{n+1} f(x_{n+1}) \\
\geq \; & (1 - \omega_{n+1}) f \left(\frac{\omega_1}{1 - \omega_{n+1}} x_1 + \cdots + \frac{\omega_n}{1 - \omega_{n+1}} x_n \right) + \omega_{n+1} f(x_{n+1}) \\
\geq \; & f \left((1 - \omega_{n+1}) \left[\frac{\omega_1}{1 - \omega_{n+1}} x_1 + \cdots + \frac{\omega_n}{1 - \omega_{n+1}} x_n \right] + \omega_{n+1} x_{n+1} \right) \\
= \; & f(\omega_1 x_1 + \cdots + \omega_{n+1} x_{n+1}).
\end{aligned}$$
□

7.2. **Power Mean Inequality.** The notion of convexity is one of the most important concepts in analysis. Jensen's Inequality is the most powerful tool in theory of inequalities. We begin with a convexity proof of The Weighted AM-GM Inequality.

Theorem 7.1. (The Weighted AM-GM Inequality) Let $\omega_1, \cdots, \omega_n > 0$ with $\omega_1 + \cdots + \omega_n = 1$. For all $x_1, \cdots, x_n > 0$, we have

$$\omega_1 \, x_1 + \cdots + \omega_n \, x_n \geq {x_1}^{\omega_1} \cdots {x_n}^{\omega_n}.$$

Proof. It is a straightforward consequence of the concavity of $\ln x$. Indeed, The Weighted Jensen's Inequality shows that

$$\ln(\omega_1 \, x_1 + \cdots + \omega_n \, x_n) \geq \omega_1 \ln(x_1) + \cdots + \omega_n \ln(x_n) = \ln({x_1}^{\omega_1} \cdots {x_n}^{\omega_n}).$$

□

The Power Mean Inequality can be proved by exploiting Jensen's inequality in two ways. We begin with two simple lemmas.

Lemma 7.1. *Let a, b, and c be positive real numbers. Let*

$$f(x) = \ln\left(\frac{a^x + b^x + c^x}{3}\right)$$

for all $x \in \mathbb{R}$. Then, we obtain $f'(0) = \ln \sqrt[3]{abc}$.

Proof. We compute

$$f'(x) = \frac{a^x \ln a + b^x \ln b + c^x \ln c}{a^x + b^x + c^x}.$$

It follows that

$$f'(0) = \frac{\ln a + \ln b + \ln c}{3} = \ln \sqrt[3]{abc}.$$

□

Lemma 7.2. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function. Suppose that f is monotone increasing on $(0, \infty)$ and monotone increasing on $(-\infty, 0)$. Then, the function f is monotone increasing on $\mathbb{R}$.*

Proof. We first show that f is monotone increasing on $[0, \infty)$. By the hypothesis, it remains to show that $f(x) \geq f(0)$ for all $x > 0$. For all $\epsilon \in (0, x)$, we have $f(x) \geq f(\epsilon)$. Since f is continuous at 0, we obtain

$$f(x) \geq \lim_{\epsilon \to 0^+} f(\epsilon) = f(0).$$

Similarly, we find that f is monotone increasing on $(-\infty, 0]$. We now show that f is monotone increasing on $\mathbb{R}$. Let x and y be real numbers with $x > y$. We want to show that $f(x) \geq f(y)$. In case $0 \notin (x, y)$, we get the result by the hypothesis. In case $x \geq 0 \geq y$, it follows that $f(x) \geq f(0) \geq f(y)$. □

Theorem 7.2. (Power Mean inequality for Three Variables) Let a, b, and c be positive real numbers. We define a function $M_{(a,b,c)} : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$M_{(a,b,c)}(0) = \sqrt[3]{abc}, \quad M_{(a,b,c)}(r) = \left(\frac{a^r + b^r + c^r}{3}\right)^{\frac{1}{r}} \quad (r \neq 0).$$

Then, $M_{(a,b,c)}$ is a monotone increasing continuous function.

First Proof. Write $M(r) = M_{(a,b,c)}(r)$. We first establish that the function M is continuous. Since M is continuous at r for all $r \neq 0$, it's enough to show that

$$\lim_{r \to 0} M(r) = \sqrt[3]{abc}.$$

Let $f(x) = \ln\left(\frac{a^x+b^x+c^x}{3}\right)$, where $x \in \mathbb{R}$. Since $f(0) = 0$, the above lemma implies that

$$\lim_{r \to 0} \frac{f(r)}{r} = \lim_{r \to 0} \frac{f(r) - f(0)}{r - 0} = f'(0) = \ln \sqrt[3]{abc}\,.$$

Since e^x is a continuous function, this means that

$$\lim_{r \to 0} M(r) = \lim_{r \to 0} e^{\frac{f(r)}{r}} = e^{\ln \sqrt[3]{abc}} = \sqrt[3]{abc}.$$

Now, we show that the function M is monotone increasing. It will be enough to establish that M is monotone increasing on $(0, \infty)$ and monotone increasing on $(-\infty, 0)$. We first show that M is monotone increasing on $(0, \infty)$. Let $x \geq y > 0$. We want to show that

$$\left(\frac{a^x + b^x + c^x}{3}\right)^{\frac{1}{x}} \geq \left(\frac{a^y + b^y + c^y}{3}\right)^{\frac{1}{y}}.$$

After the substitution $u = a^y$, $v = a^y$, $w = a^z$, it becomes

$$\left(\frac{u^{\frac{x}{y}} + v^{\frac{x}{y}} + w^{\frac{x}{y}}}{3}\right)^{\frac{1}{x}} \geq \left(\frac{u+v+w}{3}\right)^{\frac{1}{y}}.$$

Since it is homogeneous, we may normalize to $u + v + w = 3$. We are now required to show that

$$\frac{G(u) + G(v) + G(w)}{3} \geq 1,$$

where $G(t) = t^{\frac{x}{y}}$, where $t > 0$. Since $\frac{x}{y} \geq 1$, we find that G is convex. Jensen's inequality shows that

$$\frac{G(u) + G(v) + G(w)}{3} \geq G\left(\frac{u+v+w}{3}\right) = G(1) = 1.$$

Similarly, we may deduce that M is monotone increasing on $(-\infty, 0)$. □

We've learned that the convexity of $f(x) = x^\lambda \quad (\lambda \geq 1)$ implies the monotonicity of the power means. Now, we shall show that the convexity of $x \ln x$ also implies The Power Mean Inequality.

Second Proof of the Monotonicity. Write $f(x) = M_{(a,b,c)}(x)$. We use the increasing function theorem. It's enough to show that $f'(x) \geq 0$ for all $x \neq 0$. Let $x \in \mathbb{R} - \{0\}$. We compute

$$\frac{f'(x)}{f(x)} = \frac{d}{dx}(\ln f(x)) = -\frac{1}{x^2} \ln\left(\frac{a^x + b^x + c^x}{3}\right) + \frac{1}{x} \frac{\frac{1}{3}(a^x \ln a + b^x \ln b + c^x \ln c)}{\frac{1}{3}(a^x + b^x + c^x)}$$

or

$$\frac{x^2 f'(x)}{f(x)} = -\ln\left(\frac{a^x + b^x + c^x}{3}\right) + \frac{a^x \ln a^x + b^x \ln b^x + c^x \ln c^x}{a^x + b^x + c^x}.$$

To establish $f'(x) \geq 0$, we now need to establish that

$$a^x \ln a^x + b^x \ln b^x + c^x \ln c^x \geq (a^x + b^x + c^x) \ln\left(\frac{a^x + b^x + c^x}{3}\right).$$

Let us introduce a function $f : (0, \infty) \longrightarrow \mathbf{R}$ by $f(t) = t \ln t$, where $t > 0$. After the substitution $p = a^x$, $q = a^y$, $r = a^z$, it becomes

$$f(p) + f(q) + f(r) \geq 3f\left(\frac{p+q+r}{3}\right).$$

Since f is convex on $(0, \infty)$, it follows immediately from Jensen's Inequality. □

In particular, we deduce The RMS-AM-GM-HM Inequality for three variables.

Corollary 7.4. *For all positive real numbers a, b, and c, we have*

$$\sqrt{\frac{a^2+b^2+c^2}{3}} \geq \frac{a+b+c}{3} \geq \sqrt[3]{abc} \geq \frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}.$$

Proof. The Power Mean Inequality implies that

$$M_{(a,b,c)}(2) \geq M_{(a,b,c)}(1) \geq M_{(a,b,c)}(0) \geq M_{(a,b,c)}(-1).$$

□

Delta 84. [SL 2004 THA] *Let $a, b, c > 0$ and $ab + bc + ca = 1$. Prove the inequality*

$$\sqrt[3]{\frac{1}{a}+6b}+\sqrt[3]{\frac{1}{b}+6c}+\sqrt[3]{\frac{1}{c}+6a} \leq \frac{1}{abc}.$$

Delta 85. [SL 1998 RUS] *Let x, y, and z be positive real numbers such that $xyz = 1$. Prove that*

$$\frac{x^3}{(1+y)(1+z)}+\frac{y^3}{(1+z)(1+x)}+\frac{z^3}{(1+x)(1+y)} \geq \frac{3}{4}.$$

Delta 86. [LL 1992 POL] *For positive real numbers a, b, c, define*

$$A = \frac{a+b+c}{3},\ \ G = \sqrt[3]{abc},\ \ H = \frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}.$$

Prove that

$$\left(\frac{A}{G}\right)^3 \geq \frac{1}{4}+\frac{3}{4}\cdot\frac{A}{H}.$$

Using the convexity of $x \ln x$ or the convexity of x^λ $(\lambda \geq 1)$, we can also establish the monotonicity of the power means for n positive real numbers.

Theorem 7.3. (The Power Mean Inequality) Let $x_1, \cdots, x_n$ be positive real numbers. The power mean of order r is defined by

$$M_{(x_1,\cdots,x_n)}(0) = \sqrt[n]{x_1 \cdots x_n}\,,\ \ M_{(x_1,\cdots,x_n)}(r) = \left(\frac{x_1^r+\cdots+{x_n}^r}{n}\right)^{\frac{1}{r}}\quad (r \neq 0).$$

Then, the function $M_{(x_1,\cdots,x_n)} : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and monotone increasing.

Corollary 7.5. (The Geometric Mean as a Limit) Let $x_1, \cdots, x_n > 0$. Then,

$$\sqrt[n]{x_1 \cdots x_n} = \lim_{r \to 0}\left(\frac{{x_1}^r+\cdots+{x_n}^r}{n}\right)^{\frac{1}{r}}.$$

Theorem 7.4. (The RMS-AM-GM-HM Inequality) For all $x_1, \cdots, x_n > 0$, we have

$$\sqrt{\frac{{x_1}^2+\cdots+{x_n}^2}{n}} \geq \frac{x_1+\cdots+x_n}{n} \geq \sqrt[n]{x_1 \cdots x_n} \geq \frac{n}{\frac{1}{x_1}+\cdots+\frac{1}{x_n}}.$$

Delta 87. [SL 2004 IRL] *Let $a_1, \cdots, a_n$ be positive real numbers, $n > 1$. Denote by g_n their geometric mean, and by $A_1, \cdots, A_n$ the sequence of arithmetic means defined by*

$$A_k = \frac{a_1+\cdots+a_k}{k},\ k = 1, \cdots, n.$$

Let G_n be the geometric mean of $A_1, \cdots, A_n$. Prove the inequality

$$n+1 \geq \sqrt[n]{\frac{G_n}{A_n}}+\frac{g_n}{G_n}$$

and establish the cases of equality.

7.3. **Hardy-Littlewood-Pólya Inequality.** We first meet a famous inequality established by the Romanian mathematician T. Popoviciu.

Theorem 7.5. (Popoviciu's Inequality) Let $f : [a,b] \longrightarrow \mathbb{R}$ be a convex function. For all $x, y, z \in [a,b]$, we have

$$f(x)+f(y)+f(z)+3f\left(\frac{x+y+z}{3}\right) \geq 2f\left(\frac{x+y}{2}\right)+2f\left(\frac{y+z}{2}\right)+2f\left(\frac{z+x}{2}\right).$$

Proof. We break the symmetry. Since the inequality is symmetric, we may assume that $x \leq y \leq z$.

Case 1. $y \geq \frac{x+y+z}{3}$: The key idea is to make the following geometric observation:

$$\frac{z+x}{2},\ \frac{x+y}{2} \in \left[x,\ \frac{x+y+z}{3}\right].$$

It guarantees the existence of two positive weights $\lambda_1, \lambda_2 \in [0,1]$ satisfying that

$$\begin{cases} \frac{z+x}{2} = (1-\lambda_1)\,x + \lambda_1 \frac{x+y+z}{3}, \\ \frac{x+y}{2} = (1-\lambda_2)\,x + \lambda_2 \frac{x+y+z}{3}, \\ \lambda_1 + \lambda_2 = \frac{3}{2}. \end{cases}$$

Now, Jensen's inequality shows that

$$\begin{aligned} & f\left(\frac{x+y}{2}\right)+f\left(\frac{y+z}{2}\right)+f\left(\frac{z+x}{2}\right) \\ \leq\ & (1-\lambda_2)\,f(x)+\lambda_2 f\left(\frac{x+y+z}{3}\right)+\frac{f(y)+f(z)}{2}+(1-\lambda_1)\,f(x)+\lambda_1 f\left(\frac{x+y+z}{3}\right) \\ \leq\ & \frac{1}{2}\,(f(x)+f(y)+f(z))+\frac{3}{2}f\left(\frac{x+y+z}{3}\right). \end{aligned}$$

The proof of the second case uses the same idea.

Case 2. $y \leq \frac{x+y+z}{3}$: We make the following geometric observation:

$$\frac{z+x}{2},\ \frac{y+z}{2} \in \left[\frac{x+y+z}{3}, z\right].$$

It guarantees the existence of two positive weights $\mu_1, \mu_2 \in [0,1]$ satisfying that

$$\begin{cases} \frac{z+x}{2} = (1-\mu_1)\,z + \mu_1 \frac{x+y+z}{3}, \\ \frac{y+z}{2} = (1-\mu_2)\,z + \mu_2 \frac{x+y+z}{3}, \\ \mu_1 + \mu_2 = \frac{3}{2}. \end{cases}$$

Jensen's inequality implies that

$$\begin{aligned} & f\left(\frac{x+y}{2}\right)+f\left(\frac{y+z}{2}\right)+f\left(\frac{z+x}{2}\right) \\ \leq\ & \frac{f(x)+f(y)}{2}+(1-\mu_2)\,f(z)+\mu_2 f\left(\frac{x+y+z}{3}\right)+(1-\mu_1)\,f(z)+\mu_1 f\left(\frac{x+y+z}{3}\right) \\ \leq\ & \frac{1}{2}\,(f(x)+f(y)+f(z))+\frac{3}{2}f\left(\frac{x+y+z}{3}\right). \end{aligned}$$

□

Epsilon 101. *Let x, y, z be nonnegative real numbers. Then, we have*

$$3xyz + x^3 + y^3 + z^3 \geq 2\left((xy)^{\frac{3}{2}} + (yz)^{\frac{3}{2}} + (zx)^{\frac{3}{2}}\right).$$

Extending the proof of Popoviciu's Inequality, we can establish a majorization inequality.

Definition 7.2. *We say that a vector* $\mathrm{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$ *majorizes a vector* $\mathrm{y} = (y_1, \cdots, y_n) \in \mathbb{R}^n$ *if we have*

(1) $x_1 \geq \cdots \geq x_n$, $y_1 \geq \cdots \geq y_n$,
(2) $x_1 + \cdots + x_k \geq y_1 + \cdots + y_k$ *for all* $1 \leq k \leq n-1$,
(3) $x_1 + \cdots + x_n = y_1 + \cdots + y_n$.

In this case, we write $\mathrm{x} \succ \mathrm{y}$.

Theorem 7.6. (The Hardy-Littlewood-Pólya Inequality) Let $f : [a,b] \longrightarrow \mathbb{R}$ be a convex function. Suppose that $(x_1, \cdots, x_n)$ majorizes $(y_1, \cdots, y_n)$, where $x_1, \cdots, x_n, y_1, \cdots, y_n \in [a,b]$. Then, we obtain

$$f(x_1) + \cdots + f(x_n) \geq f(y_1) + \cdots + f(y_n).$$

Epsilon 102. *Let ABC be an acute triangle. Show that*

$$\cos A + \cos B + \cos C \geq 1.$$

Epsilon 103. *Let ABC be a triangle. Show that*

$$\tan^2\left(\frac{A}{4}\right) + \tan^2\left(\frac{B}{4}\right) + \tan^2\left(\frac{C}{4}\right) \leq 1.$$

Epsilon 104. *Use The Hardy-Littlewood-Pólya Inequality to deduce Popoviciu's Inequality.*

Epsilon 105. [IMO 1999/2 POL] *Let n be an integer with $n \geq 2$. (a) Determine the least constant C such that the inequality*

$$\sum_{1 \leq i < j \leq n} x_i x_j (x_i^2 + x_j^2) \leq C \left(\sum_{1 \leq i \leq n} x_i \right)^4$$

holds for all real numbers $x_1, \cdots, x_n \geq 0$.
(b) For this constant C, determine when equality holds.

It's not that I'm so smart, it's just that I stay with problems longer. - A. Einstein

8. Epsilons

God has a transfinite book with all the theorems and their best proofs

- P. Erdős

Epsilon 1. [NS] Let a and b be positive integers such that

$$a^k \mid b^{k+1}$$

for all positive integers k. Show that b is divisible by a.

Solution. Let p be a prime. Our job is to establish the equality

$$ord_p\left(b\right) \geq ord_p\left(a\right).$$

According to the condition that b^{k+1} is divisibly by a^k, we find that the inequality

$$(k+1)ord_p\left(b\right) = ord_p\left(b^{k+1}\right) \geq ord_p\left(a^k\right) = k\ ord_p\left(a\right)$$

or

$$\frac{ord_p\left(b\right)}{ord_p\left(a\right)} \geq \frac{k}{k+1}$$

holds for all positive integers k. Letting $k \to \infty$, we have the estimation

$$\frac{ord_p\left(b\right)}{ord_p\left(a\right)} \geq 1.$$

□

Epsilon 2. [IMO 1972/3 UNK] *Let m and n be arbitrary non-negative integers. Prove that*

$$\frac{(2m)!(2n)!}{m!n!(m+n)!}$$

is an integer.

Solution. We want to show that $\mathcal{L} = (2m)!(2n)!$ is divisible by $\mathcal{R} = m!n!(m+n)!$ Let p be a prime. Our job is to establish the inequality

$$ord_p\left(\mathcal{L}\right) \geq ord_p\left(\mathcal{R}\right).$$

or

$$\sum_{k=1}^{\infty}\left(\left\lfloor\frac{2m}{p^k}\right\rfloor + \left\lfloor\frac{2n}{p^k}\right\rfloor\right) \geq \sum_{k=1}^{\infty}\left(\left\lfloor\frac{m}{p^k}\right\rfloor + \left\lfloor\frac{n}{p^k}\right\rfloor + \left\lfloor\frac{m+n}{p^k}\right\rfloor\right).$$

It is an easy job to check the auxiliary inequality

$$\lfloor 2x\rfloor + \lfloor 2y\rfloor \geq \lfloor x\rfloor + \lfloor y\rfloor + \lfloor x+y\rfloor$$

holds for all real numbers x and y. $\square$

Epsilon 3. *Let* $n \in \mathbb{N}$. *Show that* $\mathcal{L}_n := lcm(1, 2, \cdots, 2n)$ *is divisible by* $\mathcal{R}_n := \binom{2n}{n} = \frac{(2n)!}{(n!)^2}$.

Solution. Let p be a prime. We want to show the inequality

$$ord_p\left(\mathcal{K}_n\right) \le ord_p\left(\mathcal{L}_n\right).$$

Now, we first compute $ord_p\left(\mathcal{K}_n\right)$.

$$ord_p\left(\mathcal{K}_n\right) = ord_p\left(\frac{(2n)!}{(n!)^2}\right) = ord_p\left(\,(2n)!\,\right) - 2ord_p\left(\,n!\,\right) = \sum_{k=1}^{\infty}\left(\left\lfloor\frac{2n}{p^k}\right\rfloor - 2\left\lfloor\frac{n}{p^k}\right\rfloor\right).$$

The key observation is that both $\frac{2n}{p^k}$ and $\frac{n}{p^k}$ vanish for all sufficiently large integer k. Let N denote the the largest integer $N \ge 0$ such that $p^N \le 2n$. The maximality of the exponent $N = ord_p\left(\mathcal{L}_n\right)$ guarantees that, whenever $k > N$, both $\frac{2n}{p^k}$ and $\frac{n}{p^k}$ are smaller than 1, so that the term $\lfloor\frac{2n}{p^k}\rfloor - 2\lfloor\frac{n}{p^k}\rfloor$ vanishes. It follows that

$$ord_p\left(\mathcal{K}_n\right) = \sum_{k=1}^{N}\left(\left\lfloor\frac{2n}{p^k}\right\rfloor - 2\left\lfloor\frac{n}{p^k}\right\rfloor\right).$$

Since $\lfloor 2x\rfloor - 2\lfloor x\rfloor$ is either 0 or 1 for all $x \in \mathbb{R}$, this gives the estimation

$$ord_p\left(\mathcal{K}_n\right) \le \sum_{k=1}^{N} 1 = N.$$

However, since $\mathcal{L}_n$ is the least common multiple of $1, \cdots, 2n$, we see that $ord_p\left(\mathcal{L}_n\right)$ is the largest integer $N \ge 0$ such that $p^N \le 2n$ □

Epsilon 4. *Let $f : \mathbb{N} \to \mathbb{R}^+$ be a function satisfying the conditions:*
(a) $f(mn) = f(m)f(n)$ for all positive integers m and n, and
(b) $f(n+1) \geq f(n)$ for all positive integers n.
Then, there is a constant $\alpha \in \mathbb{R}$ such that $f(n) = n^\alpha$ for all $n \in \mathbb{N}$.

Proof. We have $f(1) = 1$. Our job is to show that the function
$$\frac{\ln f(n)}{\ln n}$$
is constant when $n > 1$. Assume to the contrary that
$$\frac{\ln f(m)}{\ln m} > \frac{\ln f(n)}{\ln n}$$
for some positive integers $m, n > 1$. Writing $f(m) = m^x$ and $f(n) = n^y$, we have $x > y$ or
$$\frac{\ln n}{\ln m} > \frac{\ln n}{\ln m} \cdot \frac{y}{x}$$
So, we can pick a positive rational number $\frac{A}{B}$, where $A, B \in \mathbb{N}$, so that
$$\frac{\ln n}{\ln m} > \frac{A}{B} > \frac{\ln n}{\ln m} \cdot \frac{y}{x}.$$
Hence, $m^A < n^B$ and $m^{Ax} > n^{By}$. One the one hand, since f is monotone increasing, the first inequality $m^A < n^B$ means that $f\left(m^A\right) \leq f\left(n^B\right)$. On the other hand, since $f\left(m^A\right) = f(m)^A = m^{Ax}$ and $f\left(n^B\right) = f(n)^B = n^{By}$, the second inequality $m^{Ax} > n^{By}$ means that
$$f\left(m^A\right) = m^{Ax} > n^{By} = f\left(n^B\right)$$
This is a contradiction. ☐

Epsilon 5. (Putnam 1963/A2) Let $f : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function satisfying that $f(2) = 2$ and $f(mn) = f(m)f(n)$ for all relatively prime m and n. Then, f is the identity function on $\mathbb{N}$.

Proof. Since f is strictly increasing, we find that $f(n+1) \geq f(n) + 1$ for all positive integers n. It follows that $f(n+k) \geq f(n) + k$ for all positive integers n and k. We now determine $p = f(3)$. On the one hand, we obtain

$$f(18) \geq f(15) + 3 \geq f(3)f(5) + 3 \geq f(3)(f(3) + 2) + 3 = p^2 + 2p + 3.$$

On the other hand, we obtain

$$f(18) = f(2)f(9) \leq 2(f(10)-1) = 2f(2)f(5)-2 \leq 4(f(6)-1)-2 = 4f(2)f(3)-6 = 8p-6.$$

Combining these two, we deduce $p^2 + 2p + 3 \leq 8p - 6$ or $(p-3)^2 \leq 0$. So, we have $f(3) = p = 3$.

We now prove that $f\left(2^l + 1\right) = 2^l + 1$ for all positive integers l. Since $f(3) = 3$, it clearly holds for $l = 1$. Assuming that $f\left(2^l + 1\right) = 2^l + 1$ for some positive integer l, we obtain

$$f\left(2^{l+1} + 2\right) = f(2)f\left(2^l + 1\right) = 2\left(2^l + 1\right) = 2^{l+1} + 2.$$

Since f is strictly increasing, this means that $f\left(2^l + k\right) = 2^l + k$ for all $k \in \{1, \cdots, 2^l + 2\}$. In particular, we get $f\left(2^{l+1} + 1\right) = 2^{l+1} + 1$, as desired.

Now, we find that $f(n) = n$ for all positive integers n. It clearly holds for $n = 1, 2$. Let l be a fixed positive integer. We have $f\left(2^l + 1\right) = 2^l + 1$ and $f\left(2^{l+1} + 1\right) = 2^{l+1} + 1$. Since f is strictly increasing, this means that $f\left(2^l + k\right) = 2^l + k$ for all $k \in \{1, \cdots, 2^l + 1\}$. Since it holds for all positive integers l, we conclude that $f(n) = n$ for all $n \geq 3$. This completes the proof. $\square$

Epsilon 6. *Let a, b, c be positive real numbers. Prove the inequality*

$$\left(1+a^2\right)\left(1+b^2\right)\left(1+c^2\right) \geq (a+b)(b+c)(c+a).$$

Show that the equality holds if and only if $(a, b, c) = (1, 1, 1)$.

Solution. The inequality has the *symmetric* face:

$$\left(1+a^2\right)\left(1+b^2\right) \cdot \left(1+b^2\right)\left(1+c^2\right) \cdot \left(1+c^2\right)\left(1+a^2\right) \geq (a+b)^2(b+c)^2(c+a)^2.$$

Now, the symmetry of this expression gives the *right* approach. We check that, for $x, y > 0$,

$$\left(1+x^2\right)\left(1+y^2\right) \geq (x+y)^2$$

with the equality condition $xy = 1$. However, it immediately follows from the identity

$$\left(1+x^2\right)\left(1+y^2\right) - (x+y)^2 = (1-xy)^2.$$

It is easy to check that the equality in the original inequalty occurs only when $a = b = c = 1$. $\square$

Epsilon 7. (Poland 2006) Let a, b, c be positive real numbers with $ab+bc+ca = abc$. Prove that

$$\frac{a^4+b^4}{ab(a^3+b^3)} + \frac{b^4+c^4}{bc(b^3+c^3)} + \frac{c^4+a^4}{ca(c^3+a^3)} \geq 1.$$

Solution. We first notice that the constraint can be written as

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1.$$

It is now enough to establish the auxiliary inequality

$$\frac{x^4+y^4}{xy(x^3+y^3)} \geq \frac{1}{2}\left(\frac{1}{x} + \frac{1}{y}\right)$$

or

$$2\left(x^4+y^4\right) \geq \left(x^3+y^3\right)(x+y),$$

where $x, y > 0$. However, we obtain

$$2\left(x^4+y^4\right) - \left(x^3+y^3\right)(x+y) = x^4+y^4-x^3y-xy^3 = \left(x^3-y^3\right)(x-y) \geq 0.$$

$\square$

Epsilon 8. (APMO 1996) Let a, b, c be the lengths of the sides of a triangle. Prove that

$$\sqrt{a+b-c}+\sqrt{b+c-a}+\sqrt{c+a-b} \leq \sqrt{a}+\sqrt{b}+\sqrt{c}.$$

Proof. The left hand side admits the following decomposition

$$\frac{\sqrt{c+a-b}+\sqrt{a+b-c}}{2}+\frac{\sqrt{a+b-c}+\sqrt{b+c-a}}{2}+\frac{\sqrt{b+c-a}+\sqrt{c+a-b}}{2}.$$

We now use the inequality $\frac{\sqrt{x}+\sqrt{y}}{2} \leq \sqrt{\frac{x+y}{2}}$ to deduce

$$\frac{\sqrt{c+a-b}+\sqrt{a+b-c}}{2} \leq \sqrt{a},$$
$$\frac{\sqrt{a+b-c}+\sqrt{b+c-a}}{2} \leq \sqrt{b},$$
$$\frac{\sqrt{b+c-a}+\sqrt{c+a-b}}{2} \leq \sqrt{c}.$$

Adding these three inequalities, we get the result. □

Epsilon 9. *Let a, b, c be the lengths of a triangle. Show that*

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} < 2.$$

Proof. Since the inequality is symmetric in the three variables, we may assume that $a \leq b \leq c$. We obtain

$$\frac{a}{b+c} \leq \frac{a}{a+b}, \quad \frac{b}{c+a} \leq \frac{b}{a+b}, \quad \frac{c}{a+b} < 1.$$

Adding these three inequalities, we get the result. □

Epsilon 10. (USA 1980) Prove that, for all positive real numbers $a, b, c \in [0, 1]$,

$$\frac{a}{b+c+1} + \frac{b}{c+a+1} + \frac{c}{a+b+1} + (1-a)(1-b)(1-c) \leq 1.$$

Solution. Since the inequality is symmetric in the three variables, we may assume that $0 \leq a \leq b \leq c \leq 1$. Our first step is to bring the estimation

$$\frac{a}{b+c+1} + \frac{b}{c+a+1} + \frac{c}{a+b+1} \leq \frac{a}{a+b+1} + \frac{b}{a+b+1} + \frac{c}{a+b+1} \leq \frac{a+b+c}{a+b+1}.$$

It now remains to check that

$$\frac{a+b+c}{a+b+1} + (1-a)(1-b)(1-c) \leq 1.$$

or

$$(1-a)(1-b)(1-c) \leq \frac{1-c}{a+b+1}$$

or

$$(1-a)(1-b)(a+b+1) \leq 1.$$

We indeed obtain the estimation

$$(1-a)(1-b)(a+b+1) \leq (1-a)(1-b)(1+a)(1+b) = \left(1-a^2\right)\left(1-b^2\right) \leq 1.$$

$\square$

Epsilon 11. [AE, p. 186] Show that, for all $a, b, c \in [0, 1]$,

$$\frac{a}{1+bc} + \frac{b}{1+ca} + \frac{c}{1+ab} \leq 2.$$

Proof. Since the inequality is symmetric in the three variables, we may begin with the assumption $0 \leq a \geq b \geq c \leq 1$. We first give term-by-term estimation:

$$\frac{a}{1+bc} \leq \frac{a}{1+ab}, \ \frac{b}{1+ca} \leq \frac{b}{1+ab}, \ \frac{c}{1+ab} \leq \frac{1}{1+ab}.$$

Summing up these three, we reach

$$\frac{a}{1+bc} + \frac{b}{1+ca} + \frac{c}{1+ab} \leq \frac{a+b+1}{1+ab}.$$

We now want to show the inequality

$$\frac{a+b+1}{1+ab} \leq 2$$

or

$$a+b+1 \leq 2+2ab$$

or

$$a+b \leq 1+2ab.$$

However, it is immediate that $1+2ab-a-b = ab+(1-a)(1-b)$ is clearly non-negative. $\square$

Epsilon 12. [SL 2006 KOR] *Let a, b, c be the lengths of the sides of a triangle. Prove the inequality*

$$\frac{\sqrt{b+c-a}}{\sqrt{b}+\sqrt{c}-\sqrt{a}}+\frac{\sqrt{c+a-b}}{\sqrt{c}+\sqrt{a}-\sqrt{b}}+\frac{\sqrt{a+b-c}}{\sqrt{a}+\sqrt{b}-\sqrt{c}} \leq 3.$$

Solution. Since the inequality is symmetric in the three variables, we may assume that $a \geq b \geq c$. We claim that

$$\frac{\sqrt{a+b-c}}{\sqrt{a}+\sqrt{b}-\sqrt{c}} \leq 1$$

and

$$\frac{\sqrt{b+c-a}}{\sqrt{b}+\sqrt{c}-\sqrt{a}}+\frac{\sqrt{c+a-b}}{\sqrt{c}+\sqrt{a}-\sqrt{b}} \leq 2.$$

It is clear that the denominators are positive. So, the first inequality is equivalent to

$$\sqrt{a}+\sqrt{b} \geq \sqrt{a+b-c}+\sqrt{c}.$$

or

$$\left(\sqrt{a}+\sqrt{b}\right)^2 \geq \left(\sqrt{a+b-c}+\sqrt{c}\right)^2$$

or

$$\sqrt{ab} \geq \sqrt{c(a+b-c)}$$

or

$$ab \geq c(a+b-c),$$

which immediately follows from $(a-c)(b-c) \geq 0$. Now, we prove the second inequality. Setting $p=\sqrt{a}+\sqrt{b}$ and $q=\sqrt{a}-\sqrt{b}$, we obtain $a-b=pq$ and $p \geq 2\sqrt{c}$. It now becomes

$$\frac{\sqrt{c-pq}}{\sqrt{c}-q}+\frac{\sqrt{c+pq}}{\sqrt{c}+q} \leq 2.$$

We now apply The Cauchy-Schwartz Inequality to deduce

$$\begin{aligned}
\left(\frac{\sqrt{c-pq}}{\sqrt{c}-q}+\frac{\sqrt{c+pq}}{\sqrt{c}+q}\right)^2 &\leq \left(\frac{c-pq}{\sqrt{c}-q}+\frac{c+pq}{\sqrt{c}+q}\right)\left(\frac{1}{\sqrt{c}-q}+\frac{1}{\sqrt{c}+q}\right) \\
&= \frac{2\left(c\sqrt{c}-pq^2\right)}{c-q^2}\cdot\frac{2\sqrt{c}}{c-q^2} \\
&= 4\,\frac{c^2-\sqrt{c}pq^2}{\left(c-q^2\right)^2} \\
&\leq 4\,\frac{c^2-2cq^2}{\left(c-q^2\right)^2} \\
&\leq 4\,\frac{c^2-2cq^2+q^4}{\left(c-q^2\right)^2} \\
&\leq 4.
\end{aligned}$$

We find that the equality holds if and only if $a=b=c$. $\square$

Epsilon 13. *Let $f(x,y) = xy\left(x^3+y^3\right)$ for $x, y \geq 0$ with $x+y=2$. Prove the inequality*

$$f(x,y) \leq f\left(1+\frac{1}{\sqrt{3}}, 1-\frac{1}{\sqrt{3}}\right) = f\left(1-\frac{1}{\sqrt{3}}, 1+\frac{1}{\sqrt{3}}\right).$$

First Solution. We write $(x,y) = (1+\epsilon, 1-\epsilon)$ for some $\epsilon \in (-1,1)$. It follows that

$$\begin{aligned} f(x,y) &= (1+\epsilon)(1-\epsilon)\left((1+\epsilon)^3+(1-\epsilon)^3\right) \\ &= \left(1-\epsilon^2\right)\left(6\epsilon^2+2\right) \\ &= -6\left(\epsilon^2-\frac{1}{3}\right)^2+\frac{8}{3} \\ &\leq \frac{8}{3} \\ &= f\left(1\pm\frac{1}{\sqrt{3}}, 1\mp\frac{1}{\sqrt{3}}\right). \end{aligned}$$

□

Second Solution. The AM-GM Inequality gives

$$f(x,y) = xy(x+y)\left((x+y)^2-3xy\right) = 2xy(4-4xy) \leq \frac{2}{3}\left(\frac{3xy+(4-3xy)}{2}\right)^2 = \frac{8}{3}.$$

□

Epsilon 14. *Let $a, b \geq 0$ with $a + b = 1$. Prove that*

$$\sqrt{a^2 + b} + \sqrt{a + b^2} + \sqrt{1 + ab} \leq 3.$$

Show that the equality holds if and only if $(a, b) = (1, 0)$ or $(a, b) = (0, 1)$.

First Solution. We may begin with the assumption $a \geq \frac{1}{2} \geq b$. The AM-GM Inequality yields

$$2 + b \geq 1 + (1 + ab) \geq 2\sqrt{1 + ab}$$

with the equality $b = 0$. We next show that

$$3 + a \geq 4\sqrt{a^2 - a + 1}$$

or

$$(3 + a)^2 \geq 16\left(a^2 - a + 1\right)$$

or

$$(15a - 7)(1 - a) \geq 0.$$

Since we have $a \in \left[\frac{1}{2}, 1\right]$, the inequality clearly holds with the equality $a = 1$. Since we have

$$a^2 + b = a^2 - a + 1 = a + (1 - a)^2 = a + b^2$$

we conclude that

$$2\sqrt{a^2 + b} + 2\sqrt{a + b^2} + 2\sqrt{1 + ab} \leq 3 + a + (2 + b) = 6.$$

□

Epsilon 15. (USA 1981) Let ABC be a triangle. Prove that

$$\sin 3A + \sin 3B + \sin 3C \leq \frac{3\sqrt{3}}{2}.$$

Solution. We observe that the sine function is *not* cocave on $[0, 3\pi]$ and that it is negative on $(\pi, 2\pi)$. Since the inequality is symmetric in the three variables, we may assume that $A \leq B \leq C$. Observe that $A + B + C = \pi$ and that $3A, 3B, 3C \in [0, 3\pi]$. It is clear that $A \leq \frac{\pi}{3} \leq C$.

We see that either $3B \in [2\pi, 3\pi)$ or $3C \in (0, \pi)$ is impossible. In the case when $3B \in [\pi, 2\pi)$, we obtain the estimation

$$\sin 3A + \sin 3B + \sin 3C \leq 1 + 0 + 1 = 2 < \frac{3\sqrt{3}}{2}.$$

So, we may assume that $3B \in (0, \pi)$. Similarly, in the case when $3C \in [\pi, 2\pi]$, we obtain

$$\sin 3A + \sin 3B + \sin 3C \leq 1 + 1 + 0 = 2 < \frac{3\sqrt{3}}{2}.$$

Hence, we also assume $3C \in (2\pi, 3\pi)$. Now, our assmptions become $A \leq B < \frac{1}{3}\pi$ and $\frac{2}{3}\pi < C$. After the substitution $\theta = C - \frac{2}{3}\pi$, the trigonometric inequality becomes

$$\sin 3A + \sin 3B + \sin 3\theta \leq \frac{3\sqrt{3}}{2}.$$

Since $3A, 3B, 3\theta \in (0, \pi)$ and since the sine function is concave on $[0, \pi]$, Jensen's Inequality gives

$$\sin 3A + \sin 3B + \sin 3\theta \leq 3 \sin\left(\frac{3A + 3B + 3\theta}{3}\right) = 3 \sin\left(\frac{3A + 3B + 3C - 2\pi}{3}\right) = 3 \sin\left(\frac{\pi}{3}\right).$$

Under the assumption $A \leq B \leq C$, the equality occurs only when $(A, B, C) = \left(\frac{1}{9}\pi, \frac{1}{9}\pi, \frac{7}{9}\pi\right)$. $\square$

Epsilon 16. (Chebyshev's Inequality) Let $x_1, \cdots, x_n$ and $y_1, \cdots y_n$ be two monotone increasing sequences of real numbers:

$$x_1 \leq \cdots \leq x_n, \; y_1 \leq \cdots \leq y_n.$$

Then, we have the estimation

$$\sum_{i=1}^{n} x_i y_i \geq \frac{1}{n} \left(\sum_{i=1}^{n} x_i \right) \left(\sum_{i=1}^{n} y_i \right).$$

Proof. We observe that two sequences are similarly ordered in the sense that

$$(x_i - x_j)(y_i - y_j) \geq 0$$

for all $1 \leq i, j \leq n$. Now, the given inequality is an immediate consequence of the identity

$$\frac{1}{n} \sum_{i=1}^{n} x_i y_i - \frac{1}{n} \left(\sum_{i=1}^{n} x_i \right) \frac{1}{n} \left(\sum_{i=1}^{n} y_i \right) = \frac{1}{n^2} \sum_{1 \leq i, j \leq n} (x_i - x_j)(y_i - y_j).$$

□

Epsilon 17. (United Kingdom 2002) For all $a, b, c \in (0,1)$, show that

$$\frac{a}{1-a}+\frac{b}{1-b}+\frac{c}{1-c} \geq \frac{3\sqrt[3]{abc}}{1-\sqrt[3]{abc}}.$$

First Solution. Since the inequality is symmetric in the three variables, we may assume that $a \geq b \geq c$. Then, we have $\frac{1}{1-a} \geq \frac{1}{1-b} \geq \frac{1}{1-c}$. By Chebyshev's Inequality, The AM-HM Inequality and The AM-GM Inequality, we obtain

$$\begin{aligned}
\frac{a}{1-a}+\frac{b}{1-b}+\frac{c}{1-c} &\geq \frac{1}{3}(a+b+c)\left(\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}\right)\\
&\geq \frac{1}{3}(a+b+c)\left(\frac{9}{(1-a)+(1-b)+(1-c)}\right)\\
&= \frac{1}{3}\left(\frac{a+b+c}{3-(a+b+c)}\right)\\
&\geq \frac{1}{3}\cdot\frac{3\sqrt[3]{abc}}{3-3\sqrt[3]{abc}}
\end{aligned}$$

$\square$

Epsilon 18. [IMO 1995/2 RUS] *Let a, b, c be positive numbers such that $abc = 1$. Prove that*

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

First Solution. After the substitution $a = \frac{1}{x}$, $b = \frac{1}{y}$, $c = \frac{1}{z}$, we get $xyz = 1$. The inequality takes the form

$$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} \geq \frac{3}{2}.$$

Since the inequality is symmetric in the three variables, we may assume that $x \geq y \geq z$. Observe that $x^2 \geq y^2 \geq z^2$ and $\frac{1}{y+z} \geq \frac{1}{z+x} \geq \frac{1}{x+y}$. Chebyshev's Inequality and The AM-HM Inequality offer the estimation

$$\begin{aligned}
\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} &\geq \frac{1}{3}\left(x^2+y^2+z^2\right)\left(\frac{1}{y+z} + \frac{1}{z+x} + \frac{1}{x+y}\right) \\
&\geq \frac{1}{3}\left(x^2+y^2+z^2\right)\left(\frac{9}{(y+z)+(z+x)+(x+y)}\right) \\
&= \frac{3}{2} \cdot \frac{x^2+y^2+z^2}{x+y+z}.
\end{aligned}$$

Finally, we have $x^2 + y^2 + z^2 \geq \frac{1}{3}(x+y+z)^2 \geq (x+y+z)\sqrt[3]{xyz} = x+y+z$. $\square$

Epsilon 19. (Iran 1996) Let x, y, z be positive real numbers. Prove that

$$(xy + yz + zx)\left(\frac{1}{(x+y)^2} + \frac{1}{(y+z)^2} + \frac{1}{(z+x)^2}\right) \geq \frac{9}{4}.$$

First Solution. [MEK1] We assume that $x \geq y \geq z \geq 0$ and $y > 0$ (not excluding $z = 0$). Let F denote the left hand side of the inequality. We define

$$\begin{aligned}
&A = (2x + 2y - z)(x - z)(y - z) + z(x + y)^2,\\
&B = \tfrac{1}{4}x(x + y - 2z)(11x + 11y + 2z),\\
&C = (x + y)(x + z)(y + z),\\
&D = (x + y + z)(x + y - 2z) + x(y - z) + y(z - x) + (x - y)^2,\\
&E = \tfrac{1}{4}(x + y)z(x + y + 2z)^2(x + y - 2z)^2.
\end{aligned}$$

It can be verified that

$$C^2(4F - 9) = (x - y)^2[(x + y)(A + B + C) + \frac{1}{2}(x + z)(y + z)D] + E.$$

The right hand side is clearly nonnegative. It becomes an equality only for $x = y = z$ and for $x = y > 0$, $z = 0$. $\square$

Epsilon 20. (APMO 1991) Let $a_1, \cdots, a_n, b_1, \cdots, b_n$ be positive real numbers such that $a_1 + \cdots + a_n = b_1 + \cdots + b_n$. Show that

$$\frac{{a_1}^2}{a_1 + b_1} + \cdots + \frac{{a_n}^2}{a_n + b_n} \geq \frac{a_1 + \cdots + a_n}{2}.$$

First Solution. The key observation is the following identity:

$$\sum_{i=1}^{n} \frac{{a_i}^2}{a_i + b_i} = \frac{1}{2} \sum_{i=1}^{n} \frac{{a_i}^2 + {b_i}^2}{a_i + b_i},$$

which is equivalent to

$$\sum_{i=1}^{n} \frac{{a_i}^2}{a_i + b_i} = \sum_{i=1}^{n} \frac{{b_i}^2}{a_i + b_i},$$

which immediately follows from

$$\sum_{i=1}^{n} \frac{{a_i}^2}{a_i + b_i} - \sum_{i=1}^{n} \frac{{b_i}^2}{a_i + b_i} = \sum_{i=1}^{n} \frac{{a_i}^2 - {b_i}^2}{a_i + b_i} = \sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i = 0.$$

Our strategy is to establish the following *symmetric* inequality

$$\frac{1}{2} \sum_{i=1}^{n} \frac{{a_i}^2 + {b_i}^2}{a_i + b_i} \geq \frac{a_1 + \cdots + a_n + b_1 + \cdots + b_n}{4}.$$

It now remains to check the the auxiliary inequality

$$\frac{a^2 + b^2}{a + b} \geq \frac{a + b}{2},$$

where $a, b > 0$. Indeed, we have $2\left(a^2 + b^2\right) - (a + b)^2 = (a - b)^2 \geq 0$. $\square$

Epsilon 21. *Let x, y, z be positive real numbers. Show the cyclic inequality*

$$\frac{x}{2x+y} + \frac{y}{2y+z} + \frac{z}{2z+x} \le 1.$$

Solution. We first break the homogeneity. The original inequality can be rewritten as

$$\frac{1}{2+\frac{y}{x}} + \frac{1}{2+\frac{z}{y}} + \frac{1}{2+\frac{x}{z}} \le 1$$

The key idea is to employ the substitution

$$a = \frac{y}{x},\ b = \frac{z}{y},\ c = \frac{x}{z}.$$

It follows that $abc = 1$. It now admits the symmetry in the variables:

$$\frac{1}{2+a} + \frac{1}{2+b} + \frac{1}{2+c} \le 1$$

Clearing denominators, it becomes

$$(2+a)(2+b) + (2+b)(2+c) + (2+c)(2+a) \le (2+a)(2+b)(2+c)$$

or

$$12 + 4(a+b+c) + ab + bc + ca \le 8 + 4(a+b+c) + 2(ab+bc+ca) + 1$$

or

$$3 \le ab + bc + ca.$$

Applying The AM-GM Inequality, we obtain $ab + bc + ca \ge 3\,(abc)^{\frac{1}{3}} = 3.$ $\square$

Epsilon 22. *Let x, y, z be positive real numbers with $x+y+z=3$. Show the cyclic inequality*

$$\frac{x^3}{x^2+xy+y^2}+\frac{y^3}{y^2+yz+z^2}+\frac{z^3}{z^2+zx+x^2}\geq 1.$$

Proof. We begin with the observation

$$\begin{aligned}\frac{x^3-y^3}{x^2+xy+y^2}+\frac{y^3-z^3}{y^2+yz+z^2}+\frac{z^3-x^3}{z^2+zx+x^2}&\\ &= (x-y)+(y-z)+(z-x)\\ &= 0\end{aligned}$$

or

$$\frac{x^3}{x^2+xy+y^2}+\frac{y^3}{y^2+yz+z^2}+\frac{z^3}{z^2+zx+x^2}=\frac{y^3}{x^2+xy+y^2}+\frac{z^3}{y^2+yz+z^2}+\frac{x^3}{z^2+zx+x^2}.$$

Our strategy is to establish the following *symmetric* inequality

$$\frac{x^3+y^3}{x^2+xy+y^2}+\frac{y^3+z^3}{y^2+yz+z^2}+\frac{z^3+x^3}{z^2+zx+x^2}\geq 2.$$

It now remains to check the the auxiliary inequality

$$\frac{a^3+b^3}{a^2+ab+b^2}\geq\frac{a+b}{3},$$

where $a, b>0$. Indeed, we obtain the equality

$$3\left(a^3+b^3\right)-(a+b)\left(a^2+ab+b^2\right)=2(a+b)(a-b)^2.$$

We now conclude that

$$\frac{x^3+y^3}{x^2+xy+y^2}+\frac{y^3+z^3}{y^2+yz+z^2}+\frac{z^3+x^3}{z^2+zx+x^2}\geq\frac{x+y}{3}+\frac{y+z}{3}+\frac{z+x}{3}=2.$$

$\square$

Epsilon 23. [SL 1985 CAN] *Let x, y, z be positive real numbers. Show the cyclic inequality*

$$\frac{x^2}{x^2+yz}+\frac{y^2}{y^2+zx}+\frac{z^2}{z^2+xy}\leq 2.$$

First Solution. We first break the homogeneity. The original inequality can be rewritten as

$$\frac{1}{1+\frac{yz}{x^2}}+\frac{1}{1+\frac{zx}{y^2}}+\frac{1}{1+\frac{xy}{z^2}}\leq 2$$

The key idea is to employ the substitution

$$a=\frac{yz}{x^2},\ b=\frac{zx}{y^2},\ c=\frac{z^2}{xy}.$$

It then follows that $abc = 1$. It now admits the symmetry in the variables:

$$\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}\leq 2$$

Since it is symmetric in the three variables, we may break the symmetry. Let's assume $a \leq b, c$. Since it is obvious that $\frac{1}{1+a} < 1$, it is enough to check the estimation

$$\frac{1}{1+b}+\frac{1}{1+c}\leq 1$$

or equivalently

$$\frac{2+b+c}{1+b+c+bc}\leq 1$$

or equivalently

$$bc \geq 1.$$

However, it follows from $abc = 1$ and from $a \leq b, c$ that $a \leq 1$ and so that $bc \geq 1$. $\square$

Epsilon 24. [SL 1990 THA] *Let $a, b, c, d \geq 0$ with $ab+bc+cd+da=1$. show that*

$$\frac{a^3}{b+c+d}+\frac{b^3}{c+d+a}+\frac{c^3}{d+a+b}+\frac{d^3}{a+b+c} \geq \frac{1}{3}.$$

Solution. Since the constraint $ab+bc+cd+da=1$ is *not* symmetric in the variables, we *cannot* consider the case when $a \geq b \geq c \geq d$ only. We first make the observation that

$$a^2+b^2+c^2+d^2 = \frac{a^2+b^2}{2}+\frac{b^2+c^2}{2}+\frac{c^2+d^2}{2}+\frac{d^2+a^2}{2} \geq ab+bc+cd+da=1.$$

Our strategy is to establish the following result. It is *symmetric*.

Let $a, b, c, d \geq 0$ with $a^2+b^2+c^2+d^2 \geq 1$. Then, we obtain

$$\frac{a^3}{b+c+d}+\frac{b^3}{c+d+a}+\frac{c^3}{d+a+b}+\frac{d^3}{a+b+c} \geq \frac{1}{3}.$$

We now exploit the symmetry! Since everything is symmetric in the variables, we may assume that $a \geq b \geq c \geq d$. Two applications of Chebyshev's Inequality and one application of The AM-GM Inequality yield

$$\begin{aligned}
& \frac{a^3}{b+c+d}+\frac{b^3}{c+d+a}+\frac{c^3}{d+a+b}+\frac{d^3}{a+b+c} \\
\geq \; & \frac{1}{4}\left(a^3+b^3+c^3+d^3\right)\left(\frac{1}{b+c+d}+\frac{1}{c+d+a}+\frac{1}{d+a+b}+\frac{1}{a+b+c}\right) \\
\geq \; & \frac{1}{4}\left(a^3+b^3+c^3+d^3\right)\frac{4^2}{(b+c+d)+(c+d+a)+(d+a+b)+(a+b+c)} \\
\geq \; & \frac{1}{4^2}\left(a^2+b^2+c^2+d^2\right)(a+b+c+d)\frac{4^2}{3(a+b+c+d)} \\
= \; & \frac{1}{3}.
\end{aligned}$$

$\square$

Epsilon 25. [IMO 2000/2 USA] *Let a, b, c be positive numbers such that $abc = 1$. Prove that*

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1.$$

First Solution. Since $abc = 1$, we can make the substitution $a = \frac{x}{y}$, $b = \frac{y}{z}$, $c = \frac{z}{x}$ for some positive real numbers x, y, z.[15] Then, it becomes a well-known symmetric inequality:

$$\left(\frac{x}{y} - 1 + \frac{z}{y}\right)\left(\frac{y}{z} - 1 + \frac{x}{z}\right)\left(\frac{z}{x} - 1 + \frac{y}{x}\right) \leq 1$$

or

$$xyz \geq (y + z - x)(z + x - y)(x + y - z).$$

$\square$

[15]For example, take $x = 1$, $y = \frac{1}{a}$, $z = \frac{1}{ab}$.

Epsilon 26. [IMO 1983/6 USA] *Let a, b, c be the lengths of the sides of a triangle. Prove that*

$$a^2b(a-b)+b^2c(b-c)+c^2a(c-a)\geq 0.$$

First Solution. After setting $a=y+z$, $b=z+x$, $c=x+y$ for $x,y,z>0$, it becomes

$$x^3z+y^3x+z^3y\geq x^2yz+xy^2z+xyz^2$$

or

$$\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x}\geq x+y+z.$$

However, an application of The Cauchy-Schwarz Inequality gives

$$(y+z+x)\left(\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x}\right)\geq(x+y+z)^2.$$

□

Epsilon 27. [IMO 1961/2 POL] (Weitzenböck's Inequality) Let a, b, c be the lengths of a triangle with area S. Show that

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}S.$$

First Solution. Write $a = y + z$, $b = z + x$, $c = x + y$ for $x, y, z > 0$. It's equivalent to

$$((y+z)^2 + (z+x)^2 + (x+y)^2)^2 \geq 48(x+y+z)xyz,$$

which can be obtained as following :

$$((y+z)^2 + (z+x)^2 + (x+y)^2)^2 \geq 16(yz+zx+xy)^2 \geq 16 \cdot 3(xy \cdot yz + yz \cdot zx + xy \cdot yz).$$

Here, we used the well-known inequalities $p^2+q^2 \geq 2pq$ and $(p+q+r)^2 \geq 3(pq+qr+rp)$. $\square$

Epsilon 28. (Hadwiger-Finsler Inequality) For any triangle ABC with sides a, b, c and area F, the following inequality holds.

$$2ab + 2bc + 2ca - (a^2 + b^2 + c^2) \geq 4\sqrt{3}F.$$

First Proof. After the substitution $a = y + z$, $b = z + x$, $c = x + y$, where $x, y, z > 0$, it becomes

$$xy + yz + zx \geq \sqrt{3xyz(x + y + z)},$$

which follows from the identity

$$(xy + yz + zx)^2 - 3xyz(x + y + z) = \frac{(xy - yz)^2 + (yz - zx)^2 + (zx - xy)^2}{2}.$$

□

Second Proof. We now present a convexity proof. It is easy to deduce

$$\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2} = \frac{2ab + 2bc + 2ca - (a^2 + b^2 + c^{2)}}{4F}.$$

Since the function $\tan x$ is convex on $\left(0, \frac{\pi}{2}\right)$, Jensen's Inequality implies that

$$\frac{2ab + 2bc + 2ca - (a^2 + b^2 + c^{2)}}{4F} \geq 3\tan\left(\frac{\frac{A}{2} + \frac{B}{2} + \frac{C}{2}}{3}\right) = \sqrt{3}.$$

□

Epsilon 29. (Tsintsifas) Let p, q, r be positive real numbers and let a, b, c denote the sides of a triangle with area F. Then, we have

$$\frac{p}{q+r}a^2 + \frac{q}{r+p}b^2 + \frac{r}{p+q}c^2 \geq 2\sqrt{3}F.$$

Proof. (V. Pambuccian) By Hadwiger-Finsler Inequality, it suffices to show that

$$\frac{p}{q+r}a^2 + \frac{q}{r+p}b^2 + \frac{r}{p+q}c^2 \geq \frac{1}{2}(a+b+c)^2 - (a^2+b^2+c^2)$$

or

$$\left(\frac{p+q+r}{q+r}\right)a^2 + \left(\frac{p+q+r}{r+p}\right)b^2 + \left(\frac{p+q+r}{p+q}\right)c^2 \geq \frac{1}{2}(a+b+c)^2$$

or

$$((q+r)+(r+p)+(p+q))\left(\frac{1}{q+r}a^2 + \frac{1}{r+p}b^2 + \frac{1}{p+q}c^2\right) \geq (a+b+c)^2.$$

However, this is a straightforward consequence of The Cauchy-Schwarz Inequality. $\square$

Epsilon 30. (The Neuberg-Pedoe Inequality) Let a_1, b_1, c_1 denote the sides of the triangle $A_1B_1C_1$ with area F_1. Let a_2, b_2, c_2 denote the sides of the triangle $A_2B_2C_2$ with area F_2. Then, we have

$$a_1{}^2(b_2{}^2 + c_2{}^2 - a_2{}^2) + b_1{}^2(c_2{}^2 + a_2{}^2 - b_2{}^2) + c_1{}^2(a_2{}^2 + b_2{}^2 - c_2{}^2) \geq 16F_1F_2.$$

First Proof. ([LC1], Carlitz) We begin with the following lemma.

Lemma 8.1. *We have*

$$a_1{}^2(a_2{}^2 + b_2{}^2 - c_2{}^2) + b_1{}^2(b_2{}^2 + c_2{}^2 - a_2{}^2) + c_1{}^2(c_2{}^2 + a_2{}^2 - b_2{}^2) > 0.$$

Proof. Observe that it's equivalent to

$$(a_1{}^2 + b_1{}^2 + c_1{}^2)(a_2{}^2 + b_2{}^2 + c_2{}^2) > 2(a_1{}^2a_2{}^2 + b_1{}^2b_2{}^2 + c_1{}^2c_2{}^2).$$

From Heron's Formula, we find that, for $i = 1, 2$,

$$16F_i{}^2 = (a_i{}^2+b_i{}^2+c_i{}^2)^2 - 2(a_i{}^4+b_i{}^4+c_i{}^4) > 0 \quad \text{or} \quad a_i{}^2+b_i{}^2+c_i{}^2 > \sqrt{2(a_i{}^4 + b_i{}^4 + c_i{}^4)}\,.$$

The Cauchy-Schwarz Inequality implies that

$$(a_1{}^2+b_1{}^2+c_1{}^2)(a_2{}^2+b_2{}^2+c_2{}^2) > 2\sqrt{(a_1{}^4 + b_1{}^4 + c_1{}^4)(a_2{}^4 + b_2{}^4 + c_2{}^4)} \geq 2(a_1{}^2a_2{}^2+b_1{}^2b_2{}^2+c_1{}^2c_2{}^2).$$

□

By the lemma, we obtain

$$L = a_1{}^2(b_2{}^2 + c_2{}^2 - a_2{}^2) + b_1{}^2(c_2{}^2 + a_2{}^2 - b_2{}^2) + c_1{}^2(a_2{}^2 + b_2{}^2 - c_2{}^2) > 0,$$

Hence, we need to show that

$$L^2 - (16F_1{}^2)(16F_2{}^2) \geq 0.$$

One may easily check the following identity

$$L^2 - (16F_1{}^2)(16F_2{}^2) = -4(UV + VW + WU),$$

where

$$U = b_1{}^2c_2{}^2 - b_2{}^2c_1{}^2,\ V = c_1{}^2a_2{}^2 - c_2{}^2a_1{}^2 \ \text{ and } \ W = a_1{}^2b_2{}^2 - a_2{}^2b_1{}^2.$$

Using the identity

$$a_1{}^2U + b_1{}^2V + c_1{}^2W = 0 \ \text{ or } \ W = -\frac{a_1{}^2}{c_1{}^2}U - \frac{b_1{}^2}{c_1{}^2}V,$$

one may also deduce that

$$UV + VW + WU = -\frac{a_1{}^2}{c_1{}^2}\left(U - \frac{c_1{}^2 - a_1{}^2 - b_1{}^2}{2a_1{}^2}V\right)^2 - \frac{4a_1{}^2b_1{}^2 - (c_1{}^2 - a_1{}^2 - b_1{}^2)^2}{4a_1{}^2c_1{}^2}V^2.$$

It follows that

$$UV + VW + WU = -\frac{a_1{}^2}{c_1{}^2}\left(U - \frac{c_1{}^2 - a_1{}^2 - b_1{}^2}{2a_1{}^2}V\right)^2 - \frac{16F_1{}^2}{4a_1{}^2c_1{}^2}V^2 \leq 0.$$

□

Second Proof. ([LC2], Carlitz) We rewrite it in terms of $a_1, b_1, c_1, a_2, b_2, c_2$:

$$(a_1{}^2 + b_1{}^2 + c_1{}^2)(a_2{}^2 + b_2{}^2 + c_2{}^2) - 2(a_1{}^2a_2{}^2 + b_1{}^2b_2{}^2 + c_1{}^2c_2{}^2)$$

$$\geq \sqrt{\left(\left(a_1{}^2 + b_1{}^2 + c_1{}^2\right)^2 - 2(a_1{}^4 + b_1{}^4 + c_1{}^4)\right)\left(\left(a_2{}^2 + b_2{}^2 + c_2{}^2\right)^2 - 2(a_2{}^4 + b_2{}^4 + c_2{}^4)\right)}.$$

We employ the following substitutions

$$x_1 = a_1{}^2 + b_1{}^2 + c_1{}^2, x_2 = \sqrt{2}\,a_1{}^2, x_3 = \sqrt{2}\,b_1{}^2, x_4 = \sqrt{2}\,c_1{}^2,$$
$$y_1 = a_2{}^2 + b_2{}^2 + c_2{}^2, y_2 = \sqrt{2}\,a_2{}^2, y_3 = \sqrt{2}\,b_2{}^2, y_4 = \sqrt{2}\,c_2{}^2.$$

We now observe

$$x_1{}^2 > x_2{}^2 + y_3{}^2 + x_4{}^2 \text{ and } y_1{}^2 > y_2{}^2 + y_3{}^2 + y_4{}^2.$$

We now apply Aczél's inequality to get the inequality

$$x_1y_1 - x_2y_2 - x_3y_3 - x_4y_4 \geq \sqrt{(x_1{}^2 - (x_2{}^2 + y_3{}^2 + x_4{}^2))\,(y_1{}^2 - (y_2{}^2 + y_3{}^2 + y_4{}^2))}.$$

□

Epsilon 31. (Aczél's Inequality) If $a_1, \cdots, a_n, b_1, \cdots, b_n > 0$ satisfies the inequality

$$a_1{}^2 \geq a_2{}^2 + \cdots + a_n{}^2 \text{ and } b_1{}^2 \geq b_2{}^2 + \cdots + b_n{}^2,$$

then the following inequality holds.

$$a_1b_1 - (a_2b_2 + \cdots + a_nb_n) \geq \sqrt{(a_1{}^2 - (a_2{}^2 + \cdots + a_n{}^2))\left(b_1{}^2 - \left(b_2{}^2 + \cdots + b_n{}^2\right)\right)}$$

Proof. [MV] The Cauchy-Schwarz Inequality shows that

$$a_1b_1 \geq \sqrt{(a_2{}^2 + \cdots + a_n{}^2)(b_2{}^2 + \cdots + b_n{}^2)} \geq a_2b_2 + \cdots + a_nb_n.$$

Then, the above inequality is equivalent to

$$(a_1b_1 - (a_2b_2 + \cdots + a_nb_n))^2 \geq \left(a_1{}^2 - \left(a_2{}^2 + \cdots + a_n{}^2\right)\right)\left(b_1{}^2 - \left(b_2{}^2 + \cdots + b_n{}^2\right)\right).$$

In case $a_1{}^2 - (a_2{}^2 + \cdots + a_n{}^2) = 0$, it's trivial. Hence, we now assume that $a_1{}^2 - (a_2{}^2 + \cdots + a_n{}^2) > 0$. The main trick is to think of the following quadratic polynomial

$$\mathcal{P}(x) = (a_1x - b_1)^2 - \sum_{i=2}^{n}(a_ix - b_i)^2 = \left(a_1{}^2 - \sum_{i=2}^{n} a_i{}^2\right)x^2 + 2\left(a_1b_1 - \sum_{i=2}^{n} a_ib_i\right)x + \left(b_1{}^2 - \sum_{i=2}^{n} b_i{}^2\right).$$

We now observe that

$$\mathcal{P}\left(\frac{b_1}{a_1}\right) = -\sum_{i=2}^{n}\left(a_i\left(\frac{b_1}{a_1}\right) - b_i\right)^2.$$

Since $\mathcal{P}\left(\frac{b_1}{a_1}\right) \leq 0$ and since the coefficient of x^2 in the quadratic polynomial P is positive, $\mathcal{P}$ should have at least one real root. Therefore, $\mathcal{P}$ has nonnegative discriminant. It follows that

$$\left(2\left(a_1b_1 - \sum_{i=2}^{n} a_ib_i\right)\right)^2 - 4\left(a_1{}^2 - \sum_{i=2}^{n} a_i{}^2\right)\left(b_1{}^2 - \sum_{i=2}^{n} b_i{}^2\right) \geq 0.$$

□

Epsilon 32. *If A, B, C, X, Y, Z denote the magnitudes of the corresponding angles of triangles ABC, and XYZ, respectively, then*

$$\cot A\cot Y+\cot A\cot Z+\cot B\cot Z+\cot B\cot X+\cot C\cot X+\cot C\cot Y\geq 2.$$

Proof. By the Cosine Law in triangle ABC, we have $\cos A=(b^2+c^2-a^2)/2bc$. On the other hand, since $\sin A=2S/bc$, we deduce that

$$\cot A=\cos A:\sin A=\frac{b^2+c^2-a^2}{2bc}:\frac{2S}{bc}=\frac{b^2+c^2-a^2}{4S}.$$

Analogously, we have that $\cot B=(c^2+a^2-b^2)/4S$, and so,

$$\cot A+\cot B=\frac{b^2+c^2-a^2}{4S}+\frac{c^2+a^2-b^2}{4S}=\frac{2c^2}{4S}=\frac{c^2}{2S}.$$

Now since $\cot Z=(x^2+y^2-z^2)/4T$, it follows that

$$\begin{aligned}\cot A\cot Z+\cot B\cot Z &= (\cot A+\cot B)\cdot\cot Z\\ &= \frac{c^2}{2S}\cdot\frac{x^2+y^2-z^2}{4T}\\ &= \frac{c^2\left(x^2+y^2-z^2\right)}{8ST}.\end{aligned}$$

Similarly, we obtain

$$\cot B\cot X+\cot C\cot X=\frac{a^2\left(y^2+z^2-x^2\right)}{8ST},$$

and

$$\cot C\cot Y+\cot A\cot Y=\frac{b^2\left(z^2+x^2-y^2\right)}{8ST}.$$

Hence, we conclude that

$$\cot A\cot Y+\cot A\cot Z+\cot B\cot Z+\cot B\cot X+\cot C\cot X+\cot C\cot Y$$

$$\begin{aligned}&= (\cot B\cot X+\cot C\cot X)+(\ \ldots\\ &= \frac{a^2\left(y^2+z^2-x^2\right)}{8ST}+\frac{b^2\left(z^2+x \ldots\right.}{8S\ldots}\\ &= \frac{a^2\left(y^2+z^2-x^2\right)+b^2\left(z^2+x^2 \ldots\right.}{8ST}\end{aligned}$$

From the Neuberg-Pedoe Inequality, we have

$$a^2\left(y^2+z^2-x^2\right)+b^2\left(z^2+x^2-y^2\right)+c^2\left(x^2+y^2-z^2\right)\geq 16ST,$$

and so

$$\cot A\cot Y+\cot A\cot Z+\cot B\cot Z+\cot B\cot X+\cot C\cot X+\cot C\cot Y\geq 2,$$

with equality if and only if the triangles ABC and XYZ are similar. $\square$

Epsilon 33. (Vasile Cârtoaje) Let a, b, c, x, y, z be nonnegative reals. Prove the inequality

$$(ay + az + bz + bx + cx + cy)^2 \geq 4(bc + ca + ab)(yz + zx + xy),$$

with equality if and only if $a : x = b : y = c : z$.

Proof. According to the Conway substitution theorem, since a, b, c are nonnegative reals, there exists a triangle ABC with area $S = \frac{1}{2}\sqrt{bc + ca + ab}$ and with its angles A, B, C satisfying $\cot A = \frac{a}{2S}$, $\cot B = \frac{b}{2S}$, $\cot C = \frac{c}{2S}$ (note that we *cannot* denote the sidelengths of triangle ABC by a, b, c here, since a, b, c already stand for something different). Similarly, since x, y, z are nonnegative reals, there exists a triangle XYZ with area $T = \frac{1}{2}\sqrt{yz + zx + xy}$ and with its angles X, Y, Z satisfying $\cot X = \frac{x}{2T}$, $\cot Y = \frac{y}{2T}$, $\cot Z = \frac{z}{2T}$. Now, by Epsilon 32, we have

$$\cot A \cot Y + \cot A \cot Z + \cot B \cot Z + \cot B \cot X + \cot C \cot X + \cot C \cot Y \geq 2,$$

which rewrites as

$$\frac{a}{2S} \cdot \frac{y}{2T} + \frac{a}{2S} \cdot \frac{z}{2T} + \frac{b}{2S} \cdot \frac{z}{2T} + \frac{b}{2S} \cdot \frac{x}{2T} + \frac{c}{2S} \cdot \frac{x}{2T} + \frac{c}{2T} \cdot \frac{y}{2T} \geq 2,$$

and thus,

$$\begin{aligned} & ay + az + bz + bx + cx + cy \\ \geq\ & 2 \cdot 2S \cdot 2T \\ =\ & 2 \cdot 2 \cdot \frac{1}{2}\sqrt{bc + ca + ab} \cdot 2 \cdot \frac{1}{2}\sqrt{yz + zx + xy} \\ =\ & 2\sqrt{(bc + ca + ab)(yz + zx + xy)}. \end{aligned}$$

Upon squaring, this becomes

$$(ay + az + bz + bx + cx + cy)^2 \geq 4(bc + ca + ab)(yz + zx + xy).$$

□

Epsilon 34. (Walter Janous, Crux Mathematicorum) If u, v, w, x, y, z are six reals such that the terms $y+z$, $z+x$, $x+y$, $v+w$, $w+u$, $u+v$, and $vw+wu+uv$ are all nonnegative, then

$$\frac{x}{y+z} \cdot (v+w) + \frac{y}{z+x} \cdot (w+u) + \frac{z}{x+y} \cdot (u+v) \geq \sqrt{3\,(vw+wu+uv)}.$$

Proof. According to the Conway substitution theorem, since the reals $v+w$, $w+u$, $u+v$ and $vw+wu+uv$ are all nonnegative, there exists a triangle ABC with sidelengths $a = \sqrt{v+w}$, $b = \sqrt{w+u}$, $c = \sqrt{u+v}$ and area $S = \frac{1}{2}\sqrt{vw+wu+uv}$. Applying the Extended Tsintsifas Inequality to this triangle ABC and to the reals x, y, z satisfying the condition that the reals $y+z$, $z+x$, $x+y$ are all positive, we obtain

$$\frac{x}{y+z} \cdot a^2 + \frac{y}{z+x} \cdot b^2 + \frac{z}{x+y} \cdot c^2 \geq 2\sqrt{3}S,$$

which rewrites as

$$\frac{x}{y+z} \cdot \left(\sqrt{v+w}\right)^2 + \frac{y}{z+x} \cdot \left(\sqrt{w+u}\right)^2 + \frac{z}{x+y} \cdot \left(\sqrt{u+v}\right)^2 \geq 2\sqrt{3} \cdot \frac{1}{2}\sqrt{vw+wu+uv},$$

and thus,

$$\frac{x}{y+z} \cdot (v+w) + \frac{y}{z+x} \cdot (w+u) + \frac{z}{x+y} \cdot (u+v) \geq \sqrt{3\,(vw+wu+uv)}.$$

$\square$

Epsilon 35. (Tran Quang Hung) In any triangle ABC with sidelengths a, b, c, circumradius R, inradius r, and area S, we have that

$$a^2+b^2+c^2 \geq 4S\sqrt{3}+(a-b)^2+(b-c)^2+(c-a)^2+16Rr\left(\sum\cos^2\frac{A}{2}-\sum\cos\frac{B}{2}\cos\frac{C}{2}\right).$$

Proof. We know that Hadwiger-Finsler's Inequality states that

$$x^2+y^2+z^2-4T\sqrt{3} \geq (x-y)^2+(y-z)^2+(z-x)^2,$$

for any triangle XYZ with sidelengths x, y, z, and area T. Let us apply this for the triangle $XYZ = I_aI_bI_c$, where $X = I_a$, $Y = I_b$, $Z = I_c$ are the excenters of ABC. In this case, it is well-known that

$$x = 4R\cos\frac{A}{2},\ y = 4R\cos\frac{B}{2},\ z = 4R\cos\frac{C}{2},\ T = 2sR,$$

where s is the semiperimeter of triangle ABC. Therefore,

$$\begin{aligned} & 16R^2\left(\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}\right)-8s\sqrt{3} \\ \geq\ & 16R^2\left[\left(\cos^2\frac{A}{2}-\cos^2\frac{B}{2}\right)^2+\left(\cos^2\frac{B}{2}-\cos^2\frac{C}{2}\right)^2+\left(\cos^2\frac{C}{2}-\cos^2\frac{A}{2}\right)^2\right], \end{aligned}$$

which according to the well-known formulas

$$\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}},\quad \cos\frac{B}{2}=\sqrt{\frac{s(s-b)}{ca}},\quad \cos\frac{C}{2}=\sqrt{\frac{s(s-c)}{ab}},$$

easily reduces to

$$a^2+b^2+c^2 \geq 4S\sqrt{3}+(a-b)^2+(b-c)^2+(c-a)^2+16Rr\left(\sum\cos^2\frac{A}{2}-\sum\cos\frac{B}{2}\cos\frac{C}{2}\right).$$

□

Epsilon 36. *For all $\theta \in \mathbb{R}$, we have*

$$\sin(3\theta) = 4\sin\theta \sin\left(\frac{\pi}{3} + \theta\right) \sin\left(\frac{2\pi}{3} + \theta\right).$$

Proof. It follows that

$$\begin{aligned}
\sin(3\theta) &= 3\sin\theta - 4\sin^3\theta \\
&= \sin\theta\left(3\cos^2\theta - \sin^2\theta\right) \\
&= 4\sin\theta\left(\frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta\right)\left(\frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta\right) \\
&= 4\sin\theta\sin\left(\frac{\pi}{3} + \theta\right)\sin\left(\frac{2\pi}{3} + \theta\right).
\end{aligned}$$

$\square$

Epsilon 37. *For all $A, B, C \in \mathbb{R}$ with $A + B + C = 2\pi$, we have*

$$\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A \cos B \cos C = 1.$$

Proof. Our job is to show that the quadratic eqaution

$$t^2 + (2\cos B \cos C)\, t + \cos^2 B + \cos^2 C - 1 = 0$$

has a root $t = \cos A$. We find that it admits roots

$$\begin{aligned} t &= \frac{-2\cos B \cos C \pm \sqrt{4\cos^2 B \cos^2 C - 4\left(\cos^2 B + \cos^2 C - 1\right)}}{2} \\ &= -\cos B \cos C \pm \sqrt{\left(1 - \cos^2 B\right)\left(1 - \cos^2 C\right)} \\ &= -\cos B \cos C \pm |\sin B \sin C|. \end{aligned}$$

Since we have

$$-\cos B \cos C + \sin B \sin C = -\cos(B + C) = -\cos\left(\pi - A\right) = \cos A,$$

we find that $t = \cos A$ satisfies the quadratic equation, as desired. $\square$

Epsilon 38. [SL 2005 KOR] *In an acute triangle ABC, let D, E, F, P, Q, R be the feet of perpendiculars from A, B, C, A, B, C to BC, CA, AB, EF, FD, DE, respectively. Prove that*

$$p(ABC)p(PQR) \geq p(DEF)^2,$$

where $p(T)$ denotes the perimeter of triangle T .

Solution. Let's euler this problem. Let ρ be the circumradius of the triangle ABC. It's easy to show that $BC = 2\rho \sin A$ and $EF = 2\rho \sin A \cos A$. Since $DQ = 2\rho \sin C \cos B \cos A$, $DR = 2\rho \sin B \cos C \cos A$, and $\angle FDE = \pi - 2A$, the Cosine Law gives us

$$\begin{aligned} QR^2 &= DQ^2 + DR^2 - 2DQ \cdot DR \cos(\pi - 2A) \\ &= 4\rho^2 \cos^2 A \left[(\sin C \cos B)^2 + (\sin B \cos C)^2 + 2 \sin C \cos B \sin B \cos C \cos(2A)\right] \end{aligned}$$

or

$$QR = 2\rho \cos A \sqrt{f(A, B, C)},$$

where

$$f(A, B, C) = (\sin C \cos B)^2 + (\sin B \cos C)^2 + 2 \sin C \cos B \sin B \cos C \cos(2A).$$

So, what we need to attack is the following inequality:

$$\left(\sum_{\text{cyclic}} 2\rho \sin A\right)\left(\sum_{\text{cyclic}} 2\rho \cos A \sqrt{f(A, B, C)}\right) \geq \left(\sum_{\text{cyclic}} 2\rho \sin A \cos A\right)^2$$

or

$$\left(\sum_{\text{cyclic}} \sin A\right)\left(\sum_{\text{cyclic}} \cos A \sqrt{f(A, B, C)}\right) \geq \left(\sum_{\text{cyclic}} \sin A \cos A\right)^2.$$

Our job is now to find a reasonable lower bound of $\sqrt{f(A, B, C)}$. Once again, we express $f(A, B, C)$ as the sum of two squares. We observe that

$$\begin{aligned} f(A, B, C) &= (\sin C \cos B)^2 + (\sin B \cos C)^2 + 2 \sin C \cos B \sin B \cos C \cos(2A) \\ &= (\sin C \cos B + \sin B \cos C)^2 + 2 \sin C \cos B \sin B \cos C\,[-1 + \cos(2A)] \\ &= \sin^2(C + B) - 2 \sin C \cos B \sin B \cos C \cdot 2 \sin^2 A \\ &= \sin^2 A\,[1 - 4 \sin B \sin C \cos B \cos C]. \end{aligned}$$

So, we shall express $1 - 4 \sin B \sin C \cos B \cos C$ as the sum of two squares. The trick is to replace 1 with $\left(\sin^2 B + \cos^2 B\right)\left(\sin^2 C + \cos^2 C\right)$. Indeed, we get

$$\begin{aligned} 1 - 4 \sin B \sin C \cos B \cos C &= \left(\sin^2 B + \cos^2 B\right)\left(\sin^2 C + \cos^2 C\right) - 4 \sin B \sin C \cos B \cos C \\ &= (\sin B \cos C - \sin C \cos B)^2 + (\cos B \cos C - \sin B \sin C)^2 \\ &= \sin^2(B - C) + \cos^2(B + C) \\ &= \sin^2(B - C) + \cos^2 A. \end{aligned}$$

It therefore follows that

$$f(A, B, C) = \sin^2 A \left[\sin^2(B - C) + \cos^2 A\right] \geq \sin^2 A \cos^2 A$$

so that

$$\sum_{\text{cyclic}} \cos A \sqrt{f(A, B, C)} \geq \sum_{\text{cyclic}} \sin A \cos^2 A.$$

So, we can complete the proof if we establish that

$$\left(\sum_{\text{cyclic}} \sin A\right)\left(\sum_{\text{cyclic}} \sin A \cos^2 A\right) \geq \left(\sum_{\text{cyclic}} \sin A \cos A\right)^2.$$

Indeed, one sees that it's a direct consequence of The Cauchy-Schwarz Inequality

$$(p+q+r)(x+y+z) \geq (\sqrt{px}+\sqrt{qy}+\sqrt{rz})^2,$$

where p, q, r, x, y and z are positive real numbers.

□

Remark 8.1. *Alternatively, one may obtain another lower bound of $f(A,B,C)$:*

$$\begin{aligned}
f(A,B,C) &= (\sin C\cos B)^2 + (\sin B\cos C)^2 + 2\sin C\cos B\sin B\cos C\cos(2A)\\
&= (\sin C\cos B - \sin B\cos C)^2 + 2\sin C\cos B\sin B\cos C\,[1+\cos(2A)]\\
&= \sin^2(B-C) + 2\frac{\sin(2B)}{2}\cdot\frac{\sin(2C)}{2}\cdot 2\cos^2 A\\
&\geq \cos^2 A\sin(2B)\sin(2C).
\end{aligned}$$

Then, we can use this to offer a lower bound of the perimeter of triangle PQR:

$$p(PQR) = \sum_{\text{cyclic}} 2\rho\cos A\sqrt{f(A,B,C)} \geq \sum_{\text{cyclic}} 2\rho\cos^2 A\sqrt{\sin 2B\sin 2C}$$

So, one may consider the following inequality:

$$p(ABC)\sum_{\text{cyclic}} 2\rho\cos^2 A\sqrt{\sin 2B\sin 2C} \geq p(DEF)^2$$

or

$$\left(2\rho\sum_{\text{cyclic}}\sin A\right)\left(\sum_{\text{cyclic}} 2\rho\cos^2 A\sqrt{\sin 2B\sin 2C}\right) \geq \left(2\rho\sum_{\text{cyclic}}\sin A\cos A\right)^2.$$

or

$$\left(\sum_{\text{cyclic}}\sin A\right)\left(\sum_{\text{cyclic}}\cos^2 A\sqrt{\sin 2B\sin 2C}\right) \geq \left(\sum_{\text{cyclic}}\sin A\cos A\right)^2.$$

However, it turned out that this doesn't hold. Disprove this!

Epsilon 39. [IMO 2001/1 KOR] *Let ABC be an acute-angled triangle with O as its circumcenter. Let P on line BC be the foot of the altitude from A. Assume that $\angle BCA \geq \angle ABC + 30^\circ$. Prove that $\angle CAB + \angle COP < 90^\circ$.*

Solution. The angle inequality $\angle CAB+\angle COP < 90^\circ$ can be written as $\angle COP < \angle PCO$. This can be shown if we establish the length inequality $OP > PC$. Since the power of P with respect to the circumcircle of ABC is $OP^2 = R^2 - BP \cdot PC$, where R is the circumradius of the triangle ABC, it becomes $R^2 - BP \cdot PC > PC^2$ or $R^2 > BC \cdot PC$. We euler this. It's an easy job to get $BC = 2R\sin A$ and $PC = 2R\sin B\cos C$. Hence, we show the inequality $R^2 > 2R\sin A \cdot 2R\sin B\cos C$ or $\sin A\sin B\cos C < \frac{1}{4}$. Since $\sin A < 1$, it suffices to show that $\sin A\sin B\cos C < \frac{1}{4}$. Finally, we use the angle condition $\angle C \geq \angle B + 30^\circ$ to obtain the trigonometric inequality

$$\sin B\cos C = \frac{\sin(B+C) - \sin(C-B)}{2} \leq \frac{1-\sin(C-B)}{2} \leq \frac{1-\sin 30^\circ}{2} = \frac{1}{4}.$$

□

Epsilon 40. [IMO 1961/2 POL] (Weitzenböck's Inequality) Let a, b, c be the lengths of a triangle with area S. Show that

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}S.$$

Second Proof. [AE, p.171] Let ABC be a triangle with sides $BC = a$, $CA = b$ and $AB = c$. After taking the point P on the same side of BC as the vertex A so that $\triangle PBC$ is equilateral, we use The Cosine Law to deduce the geometric identity

$$\begin{aligned} AP^2 &= b^2 + c^2 - 2bc\cos\left|C - \frac{\pi}{6}\right| \\ &= b^2 + c^2 - 2bc\cos\left(C - \frac{\pi}{6}\right) \\ &= b^2 + c^2 - bc\cos C - \sqrt{3}bc\sin C \\ &= b^2 + c^2 - \frac{b^2 + c^2 - a^2}{2} - 2\sqrt{3}K \end{aligned}$$

which implies the geometric inequality

$$b^2 + c^2 - \frac{b^2 + c^2 - a^2}{2} \geq 2\sqrt{3}K$$

or equivalently

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}S.$$

□

Epsilon 41. (The Neuberg-Pedoe Inequality) Let a_1, b_1, c_1 denote the sides of the triangle $A_1B_1C_1$ with area F_1. Let a_2, b_2, c_2 denote the sides of the triangle $A_2B_2C_2$ with area F_2. Then, we have

$$a_1{}^2(b_2{}^2 + c_2{}^2 - a_2{}^2) + b_1{}^2(c_2{}^2 + a_2{}^2 - b_2{}^2) + c_1{}^2(a_2{}^2 + b_2{}^2 - c_2{}^2) \geq 16F_1F_2.$$

Third Proof. [DP2] We take the point P on the same side of B_1C_1 as the vertex A_1 so that $\triangle PB_1C_1 \sim \triangle A_2B_2C_2$. Now, we use The Cosine Law to deduce the geometric identity

$$\begin{aligned}
& a_2{}^2 A_1P^2 \\
= \quad & a_2{}^2 b_1{}^2 + b_2{}^2 a_1{}^2 - 2a_1a_2b_1b_2 \cos|C_1 - C_2| \\
= \quad & a_2{}^2 b_1{}^2 + b_2{}^2 a_1{}^2 - 2a_1a_2b_1b_2 \cos(C_1 - C_2) \\
= \quad & a_2{}^2 b_1{}^2 + b_2{}^2 a_1{}^2 - \frac{1}{2}(2a_1b_1\cos C_1)(2a_2b_2\cos C_2) - 8\left(\frac{1}{2}a_1b_1\sin C_1\right)\left(\frac{1}{2}a_2b_2\sin C_2\right) \\
= \quad & a_2{}^2 b_1{}^2 + b_2{}^2 a_1{}^2 - \frac{1}{2}\left(a_1{}^2 + b_1{}^2 - c_1{}^2\right)\left(a_1{}^2 + b_1{}^2 - c_1{}^2\right) - 8F_1F_2,
\end{aligned}$$

which implies the geometric inequality

$$a_2{}^2 b_1{}^2 + b_2{}^2 a_1{}^2 - \frac{1}{2}\left(a_1{}^2 + b_1{}^2 - c_1{}^2\right)\left(a_1{}^2 + b_1{}^2 - c_1{}^2\right) \geq 8F_1F_2$$

or equivalently

$$a_1{}^2(b_2{}^2 + c_2{}^2 - a_2{}^2) + b_1{}^2(c_2{}^2 + a_2{}^2 - b_2{}^2) + c_1{}^2(a_2{}^2 + b_2{}^2 - c_2{}^2) \geq 16F_1F_2.$$

$\square$

Epsilon 42. (Barrow's Inequality) Let P be an interior point of a triangle ABC and let U, V, W be the points where the bisectors of angles BPC, CPA, APB cut the sides BC,CA,AB respectively. Then, we have

$$PA + PB + PC \geq 2(PU + PV + PW).$$

Proof. ([MB] and [AK]) Let $d_1 = PA$, $d_2 = PB$, $d_3 = PC$, $l_1 = PU$, $l_2 = PV$, $l_3 = PW$, $2\theta_1 = \angle BPC$, $2\theta_2 = \angle CPA$, and $2\theta_3 = \angle APB$. We need to show that $d_1 + d_2 + d_3 \geq 2(l_1 + l_2 + l_3)$. It's easy to deduce the following identities

$$l_1 = \frac{2d_2d_3}{d_2 + d_3}\cos\theta_1,\ l_2 = \frac{2d_3d_1}{d_3 + d_1}\cos\theta_2,\ \text{ and }\ l_3 = \frac{2d_1d_2}{d_1 + d_2}\cos\theta_3,$$

It now follows that

$$l_1 + l_2 + l_3 \leq \sqrt{d_2d_3}\cos\theta_1 + \sqrt{d_3d_1}\cos\theta_2 + \sqrt{d_1d_2}\cos\theta_3 \leq \frac{1}{2}\left(d_1 + d_2 + d_3\right).$$

□

Epsilon 43. ([AK], Abi-Khuzam) Let $x_1, \cdots, x_4$ be positive real numbers. Let $\theta_1, \cdots, \theta_4$ be real numbers such that $\theta_1 + \cdots + \theta_4 = \pi$. Then, we have

$$x_1 \cos\theta_1 + x_2 \cos\theta_2 + x_3 \cos\theta_3 + x_4 \cos\theta_4 \leq \sqrt{\frac{(x_1x_2 + x_3x_4)(x_1x_3 + x_2x_4)(x_1x_4 + x_2x_3)}{x_1x_2x_3x_4}}.$$

Proof. Let $p = \frac{{x_1}^2+{x_2}^2}{2x_1x_2} + \frac{{x_3}^2+{x_4}^2}{2x_3x_4}$ $q = \frac{x_1x_2+x_3x_4}{2}$ and $\lambda = \sqrt{\frac{p}{q}}$. In the view of $\theta_1 + \theta_2 + (\theta_3 + \theta_4) = \pi$ and $\theta_3 + \theta_4 + (\theta_1 + \theta_2) = \pi$, we have

$$x_1 \cos\theta_1 + x_2 \cos\theta_2 + \lambda \cos(\theta_3 + \theta_4) \leq p\lambda = \sqrt{pq},$$

and

$$x_3 \cos\theta_3 + x_4 \cos\theta_4 + \lambda \cos(\theta_1 + \theta_2) \leq \frac{q}{\lambda} = \sqrt{pq}.$$

Since $\cos(\theta_3 + \theta_4) + \cos(\theta_1 + \theta_2) = 0$, adding these two above inequalities yields

$$x_1 \cos\theta_1 + x_2 \cos\theta_2 + x_3 \cos\theta_3 + x_4 \cos\theta_4 \leq 2\sqrt{pq} = \sqrt{\frac{(x_1x_2 + x_3x_4)(x_1x_3 + x_2x_4)(x_1x_4 + x_2x_3)}{x_1x_2x_3x_4}}.$$

□

Epsilon 44. [IMO 1991/5 FRA] *Let ABC be a triangle and P an interior point in ABC. Show that at least one of the angles $\angle PAB$, $\angle PBC$, $\angle PCA$ is less than or equal to 30°.*

First Proof. Set $A_1 = A$, $A_2 = B$, $A_3 = C$, $A_4 = A$ and write $\angle PA_iA_{i+1} = \theta_i$. Let H_1, H_2, H_3 denote the feet of perpendiculars from P to the sides BC, CA, AB, respectively. Now, we assume to the contrary that $\theta_1, \theta_2, \theta_3 > \frac{\pi}{6}$. Since the angle sum of a triangle is 180°, it is immediate that $\theta_1, \theta_2, \theta_3 < \frac{5\pi}{6}$. Hence,

$$\frac{PH_i}{PA_{i+1}} = \sin\theta_i > \frac{1}{2},$$

for all $i = 1, 2, 3$. We now find that

$$2\left(PH_1 + PH_2 + PH_3\right) > PA_2 + PA_3 + PA_1,$$

which contradicts for The Erdős-Mordell Theorem. $\square$

Epsilon 45. *Any triangle has the same Brocard angles.*

Proof. More strongly, we show that the isogonal conjugate of the first Brocard point is the second Brocard point. Let Ω_1, Ω_2 denote the Brocard points of a triangle ABC, respectively. Let ω_1, ω_2 be the corresponding Brocard angles. Take the isogonal conjugate point Ω of Ω_1. Then, by the definition of isogonal conjugate point, we find that

$$\angle \Omega BA = \angle \Omega_2 CB = \angle \Omega_2 AC = \omega_1.$$

Hence, we see that the interior point Ω is the the second Brocard point of ABC. By the *uniqueness* of the second Brocard point of ABC, we see that $\Omega = \Omega_2$ and that $\omega_1 = \omega_2$. $\square$

Epsilon 46. *The Brocard angle ω of the triangle ABC satisfies*

$$\cot\omega = \cot A + \cot B + \cot C.$$

Proof. Let Ω denote the first Brocard point of ABC. We only prove it in the case when ABC is acute. Let AH, PQ denote the altitude from A, Q, respectively. Both angles $\angle B$ and $\angle C$ are acute, the point H lies on the interior side of BC. Let $P \neq \Omega$ be the intersubsection point of the circumcircle of triangle ΩCA with ray $B\Omega$. Since $\angle APB = \angle AP\Omega = \angle AC\Omega = \omega = \angle \Omega BC = \angle PBC$, we find that AP is parallel to BC so that $AH = PQ$. Since $\angle A$ is acute or since $\angle PCB = \angle PBA + \angle C = \angle B + \angle C = 180^\circ - \angle A$ is obtuse, we see that the point H lies on the outside of side BC. Since the four points B, H, C, Q are collinear in this order, we have $BQ = BH + HC + CQ$. It thus follows that

$$\cot\omega = \frac{BQ}{PQ} = \frac{BH}{AH} + \frac{HC}{AH} + \frac{CQ}{PQ} = \cot A + \cot B + \cot C.$$

$\square$

Epsilon 47. (The Trigonometric Version of Ceva's Theorem) For an interior point P of a triangle $A_1A_2A_3$, we write

$$\begin{aligned}\angle A_3A_1A_2 = \alpha_1,\ \angle PA_1A_2 = \vartheta_1,\ \angle PA_1A_3 = \theta_1,\\ \angle A_1A_2A_3 = \alpha_2,\ \angle PA_2A_3 = \vartheta_2,\ \angle PA_2A_1 = \theta_2,\\ \angle A_2A_3A_1 = \alpha_3,\ \angle PA_3A_1 = \vartheta_3,\ \angle PA_3A_2 = \theta_3.\end{aligned}$$

Then, we find a *hidden* symmetry:

$$\frac{\sin\vartheta_1}{\sin\theta_1}\cdot\frac{\sin\vartheta_2}{\sin\theta_2}\cdot\frac{\sin\vartheta_3}{\sin\theta_3} = 1$$

or equivalently

$$\frac{1}{\sin\alpha_1\sin\alpha_2\sin\alpha_3} = [\cot\vartheta_1 - \cot\alpha_1]\,[\cot\vartheta_2 - \cot\alpha_2]\,[\cot\vartheta_3 - \cot\alpha_3]\,.$$

Proof. Applying The Sine Law, we have

$$\frac{\sin\vartheta_1}{\sin\theta_1} = \frac{PA_2}{PA_1},\ \frac{\sin\vartheta_2}{\sin\theta_2} = \frac{PA_3}{PA_2},\ \frac{\sin\vartheta_3}{\sin\theta_3} = \frac{PA_1}{PA_3}.$$

It follows that

$$\frac{\sin\vartheta_1}{\sin\theta_1}\cdot\frac{\sin\vartheta_2}{\sin\theta_2}\cdot\frac{\sin\vartheta_3}{\sin\theta_3} = \frac{PA_2}{PA_1}\cdot\frac{PA_3}{PA_2}\cdot\frac{PA_1}{PA_3} = 1.$$

We now observe that, for $i = 1, 2, 3$,

$$\cot\vartheta_i - \cot\alpha_i = \frac{\cos\vartheta_i}{\sin\vartheta_i} - \frac{\cos\alpha_i}{\sin\alpha_i} = \frac{\sin(\alpha_i - \vartheta_i)}{\sin\alpha_i\sin\vartheta_i} = \frac{\sin\theta_i}{\sin\alpha_i\sin\vartheta_i}.$$

It therefore follows that

$$\begin{aligned}&[\cot\vartheta_1 - \cot\alpha_1]\,[\cot\vartheta_2 - \cot\alpha_2]\,[\cot\vartheta_3 - \cot\alpha_3]\\ =\ & \frac{\sin\theta_1}{\sin\alpha_1\sin\vartheta_1}\cdot\frac{\sin\theta_2}{\sin\alpha_2\sin\vartheta_2}\cdot\frac{\sin\theta_3}{\sin\alpha_3\sin\vartheta_3}\\ =\ & \frac{1}{\sin\alpha_1\sin\alpha_2\sin\alpha_3}\cdot\frac{\sin\theta_1}{\sin\vartheta_1}\cdot\frac{\sin\theta_2}{\sin\vartheta_2}\cdot\frac{\sin\theta_3}{\sin\vartheta_3}\\ =\ & \frac{1}{\sin\alpha_1\sin\alpha_2\sin\alpha_3}.\end{aligned}$$

□

Epsilon 48. *Let P be an interior point of a triangle ABC. Show that*

$$\cot(\angle PAB) + \cot(\angle PBC) + \cot(\angle PCA) \geq 3\sqrt{3}.$$

Proof. Set $A_1 = A$, $A_2 = B$, $A_3 = C$, $A_4 = A$ and write $\angle A_i = \alpha_i$ and $\angle PA_iA_{i+1} = \vartheta_i$ for $i = 1, 2, 3$. Our job is to establish the inequality

$$\cot\vartheta_1 + \cot\vartheta_2 + \cot\vartheta_3 \geq 3\sqrt{3}.$$

We begin with The Trigonometric Version of Ceva's Theorem

$$\frac{1}{\sin\alpha_1 \sin\alpha_2 \sin\alpha_3} = [\cot\vartheta_1 - \cot\alpha_1][\cot\vartheta_2 - \cot\alpha_2][\cot\vartheta_3 - \cot\alpha_3].$$

We first apply The AM-GM Inequality and Jensen's Inequality to deduce

$$\sin\alpha_1 \sin\alpha_2 \sin\alpha_3 \leq \left(\frac{\sin\alpha_1 + \sin\alpha_2 + \sin\alpha_3}{3}\right)^3 \leq \sin^3\left(\frac{\alpha_1 + \alpha_2 + \alpha_3}{3}\right) = \left(\frac{\sqrt{3}}{2}\right)^3$$

or

$$\left(\frac{2}{\sqrt{3}}\right)^3 \leq [\cot\vartheta_1 - \cot\alpha_1][\cot\vartheta_2 - \cot\alpha_2][\cot\vartheta_3 - \cot\alpha_3].$$

Since $\vartheta_i \in (0, \alpha_i)$, the monotonicity of the cotangent function shows that $\cot\alpha_i - \cot\vartheta_i$ is positive. Hence, by The AM-GM Inequality, the above inequality guarantees that

$$\begin{aligned}
\frac{2}{\sqrt{3}} &\leq \sqrt[3]{[\cot\vartheta_1 - \cot\alpha_1][\cot\vartheta_2 - \cot\alpha_2][\cot\vartheta_3 - \cot\alpha_3]} \\
&\leq \frac{[\cot\vartheta_1 - \cot\alpha_1] + [\cot\vartheta_2 - \cot\alpha_2] + [\cot\vartheta_3 - \cot\alpha_3]}{3} \\
&= \frac{[\cot\vartheta_1 + \cot\vartheta_2 + \cot\vartheta_3] - [\cot\alpha_1 + \cot\alpha_2 + \cot\alpha_3]}{3}
\end{aligned}$$

or

$$\cot\vartheta_1 + \cot\vartheta_2 + \cot\vartheta_3 \geq \cot\alpha_1 + \cot\alpha_2 + \cot\alpha_3 + 2\sqrt{3}.$$

Since we know $\cot\alpha_1 + \cot\alpha_2 + \cot\alpha_3 \geq \sqrt{3}$, we get the desired inequality. □

Epsilon 49. [IMO 1961/2 POL] (Weitzenböck's Inequality) Let a, b, c be the lengths of a triangle with area S. Show that

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}S.$$

Fourth Proof. ([RW], R. Weitzenböck) Let ABC be a triangle with sides a, b, and c. To euler it, we toss the picture on the real plane $\mathbb{R}^2$ with the coordinates $A(\alpha, \beta)$, $B\left(-\frac{a}{2}, 0\right)$ and $C\left(\frac{a}{2}, 0\right)$. Now, we obtain

$$\left(a^2 + b^2 + c^2\right)^2 - \left(4\sqrt{3}S\right)^2 = \left(\frac{3}{2}a^2 + \left(\alpha^2 - \beta^2\right)\right)^2 + 16\alpha^2\beta^2 \geq 0.$$

$\square$

Epsilon 50. (The Neuberg-Pedoe Inequality) Let a_1, b_1, c_1 denote the sides of the triangle $A_1B_1C_1$ with area F_1. Let a_2, b_2, c_2 denote the sides of the triangle $A_2B_2C_2$ with area F_2. Then, we have

$$a_1{}^2(b_2{}^2 + c_2{}^2 - a_2{}^2) + b_1{}^2(c_2{}^2 + a_2{}^2 - b_2{}^2) + c_1{}^2(a_2{}^2 + b_2{}^2 - c_2{}^2) \geq 16F_1F_2.$$

Fourth Proof. (By a participant from KMO[16] summer program.) We toss $\triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$ onto the real plane $\mathbb{R}^2$:

$$A_1(0, p_1),\ B_1(p_2, 0),\ C_1(p_3, 0),\ A_2(0, q_1),\ B_2(q_2, 0),\ \text{and } C_2(q_3, 0).$$

It therefore follows from the inequality $x^2 + y^2 \geq 2|xy|$ that

$$\begin{aligned}
& a_1{}^2(b_2{}^2 + c_2{}^2 - a_2{}^2) + b_1{}^2(c_2{}^2 + a_2{}^2 - b_2{}^2) + c_1{}^2(a_2{}^2 + b_2{}^2 - c_2{}^2) \\
= \ & (p_3 - p_2)^2(2q_1{}^2 + 2q_1q_2) + (p_1{}^2 + p_3{}^2)(2q_2{}^2 - 2q_2q_3) + (p_1{}^2 + p_2{}^2)(2q_3{}^2 - 2q_2q_3) \\
= \ & 2(p_3 - p_2)^2 q_1{}^2 + 2(q_3 - q_2)^2 p_1{}^2 + 2(p_3q_2 - p_2q_3)^2 \\
\geq \ & 2((p_3 - p_2)q_1)^2 + 2((q_3 - q_2)p_1)^2 \\
\geq \ & 4|(p_3 - p_2)q_1| \cdot |(q_3 - q_2)p_1| \\
= \ & 16F_1F_2\ .
\end{aligned}$$

□

[16]Korean Mathematical Olympiads

Epsilon 51. (USA 2003) Let ABC be a triangle. A circle passing through A and B intersects the segments AC and BC at D and E, respectively. Lines AB and DE intersect at F, while lines BD and CF intersect at M. Prove that $MF = MC$ if and only if $MB \cdot MD = MC^2$.

Solution. (Darij Grinberg) By Ceva's theorem, applied to the triangle BCF and the concurrent cevians BM, CA and FE (in fact, these cevians concur at the point D), we have

$$\frac{MF}{MC} \cdot \frac{EC}{EB} \cdot \frac{AB}{AF} = 1.$$

Hence, $\frac{MF}{MC} = \frac{AF}{AB} \cdot \frac{EB}{EC} = \frac{AF}{AB} : \frac{EC}{EB}$. Thus, $MF = MC$ holds if and only if $\frac{AF}{AB} = \frac{EC}{EB}$. But by Thales' theorem, $\frac{AF}{AB} = \frac{EC}{EB}$ is equivalent to $AE|FC$, and obviously we have $AE|FC$ if and only if $\angle EAC = \angle ACF$. Now, since the points A, B, D and E lie on one circle, we have that $\angle EAD = \angle EBD$, what rewrites as $\angle EAC = \angle CBM$. On other hand, we trivially have that $\angle ACF = \angle DCM$. Thus, $\angle EAC = \angle ACF$ if and only if $\angle CBM = \angle DCM$. Now, as it is clear that $\angle CMB = \angle DMC$, we have $\angle CBM = \angle DCM$ if and only if the triangles CMB and DMC are similar. But, the triangles CMB and DMC are similar if and only if $\frac{MB}{MC} = \frac{MC}{MD}$. This is finally equivalent to $MB \cdot MD = MC^2$, and so, by combining all these equivalences, the conclusion follows. $\square$

Epsilon 52. [TD] Let P be an arbitrary point in the plane of a triangle ABC with the centroid G. Show the following inequalities

(1) $\overline{BC} \cdot \overline{PB} \cdot \overline{PC} + \overline{AB} \cdot \overline{PA} \cdot \overline{PB} + \overline{CA} \cdot \overline{PC} \cdot \overline{PA} \geq \overline{BC} \cdot \overline{CA} \cdot \overline{AB}$ and

(2) $\overline{PA}^3 \cdot \overline{BC} + \overline{PB}^3 \cdot \overline{CA} + \overline{PC}^3 \cdot \overline{AB} \geq 3\overline{PG} \cdot \overline{BC} \cdot \overline{CA} \cdot \overline{AB}$.

Solution. We only check the first inequality. We regard A, B, C, P as complex numbers and assume that P corresponds to 0. We're required to prove that

$$|(B-C)BC| + |(A-B)AB| + |(C-A)CA| \geq |(B-C)(C-A)(A-B)|.$$

It remains to apply The Triangle Inequality to the algebraic identity

$$(B-C)BC + (A-B)AB + (C-A)CA = -(B-C)(C-A)(A-B).$$

$\square$

Epsilon 53. (The Neuberg-Pedoe Inequality) Let a_1, b_1, c_1 denote the sides of the triangle $A_1B_1C_1$ with area F_1. Let a_2, b_2, c_2 denote the sides of the triangle $A_2B_2C_2$ with area F_2. Then, we have

$$a_1{}^2(b_2{}^2 + c_2{}^2 - a_2{}^2) + b_1{}^2(c_2{}^2 + a_2{}^2 - b_2{}^2) + c_1{}^2(a_2{}^2 + b_2{}^2 - c_2{}^2) \geq 16F_1F_2.$$

Fifth Proof. ([GC], G. Chang) We regard A, B, C, A', B', C' as complex numbers and assume that C corresponds to 0. Rewriting the both sides in the inequality in terms of complex numbers, we get

$$\begin{aligned} & a_1{}^2(b_2{}^2 + c_2{}^2 - a_2{}^2) + b_1{}^2(c_2{}^2 + a_2{}^2 - b_2{}^2) + c_1{}^2(a_2{}^2 + b_2{}^2 - c_2{}^2) \\ = \quad & 2\left(|A'|^2|B|^2 + |A|^2|B'|^2\right) - \left(A\overline{B} + \overline{A}B\right)\left(A'\overline{B'} + \overline{A'}B\right) \end{aligned}$$

and

$$16F_1F_2 = \pm\left(\overline{A}B - A\overline{B}\right)\left(A'\overline{B'} + \overline{A'}B'\right),$$

where the sign begin chose to make the right hand positive. According to whether the triangle ABC and the triangle $A'B'C'$ have the same orientation or not, we obtain either

$$a_1{}^2(b_2{}^2 + c_2{}^2 - a_2{}^2) + b_1{}^2(c_2{}^2 + a_2{}^2 - b_2{}^2) + c_1{}^2(a_2{}^2 + b_2{}^2 - c_2{}^2) - 16F_1F_2 = 2\left|AB' - A'B\right|^2$$

or

$$a_1{}^2(b_2{}^2 + c_2{}^2 - a_2{}^2) + b_1{}^2(c_2{}^2 + a_2{}^2 - b_2{}^2) + c_1{}^2(a_2{}^2 + b_2{}^2 - c_2{}^2) - 16F_1F_2 = 2\left|A\overline{B'} - \overline{A'}B\right|^2.$$

This completes the proof. □

Epsilon 54. [SL 2002 KOR] *Let ABC be a triangle for which there exists an interior point F such that $\angle AFB = \angle BFC = \angle CFA$. Let the lines BF and CF meet the sides AC and AB at D and E, respectively. Prove that $\overline{AB} + \overline{AC} \geq 4\overline{DE}$.*

Solution. Let $\overline{AF} = x, \overline{BF} = y, \overline{CF} = z$ and let $\omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$. We can toss the pictures on $\mathbb{C}$ so that the points F, A, B, C, D, and E are represented by the complex numbers 0, x, $y\omega$, $z\omega^2$, d, and e. It's an easy exercise to establish that $\overline{DF} = \frac{xz}{x+z}$ and $\overline{EF} = \frac{xy}{x+y}$. This means that $d = -\frac{xz}{x+z}\omega$ and $e = -\frac{xy}{x+y}\omega$. We're now required to prove that

$$|x - y\omega| + |z\omega^2 - x| \geq 4\left|\frac{-zx}{z+x}\omega + \frac{xy}{x+y}\omega^2\right|.$$

Since $|\omega| = 1$ and $\omega^3 = 1$, we have $|z\omega^2 - x| = |\omega(z\omega^2 - x)| = |z - x\omega|$. Therefore, we need to prove

$$|x - y\omega| + |z - x\omega| \geq \left|\frac{4zx}{z+x} - \frac{4xy}{x+y}\omega\right|.$$

More strongly, we establish that $|(x - y\omega) + (z - x\omega)| \geq \left|\frac{4zx}{z+x} - \frac{4xy}{x+y}\omega\right|$ or $|p - q\omega| \geq |r - s\omega|$, where $p = z + x$, $q = y + x$, $r = \frac{4zx}{z+x}$ and $s = \frac{4xy}{x+y}$. It's clear that $p \geq r > 0$ and $q \geq s > 0$. It follows that

$$|p - q\omega|^2 - |r - s\omega|^2 = (p - q\omega)\overline{(p - q\omega)} - (r - s\omega)\overline{(r - s\omega)} = (p^2 - r^2) + (pq - rs) + (q^2 - s^2) \geq 0.$$

It's easy to check that the equality holds if and only if $\triangle ABC$ is equilateral. $\square$

Epsilon 55. (APMO 2004/5) Prove that, for all positive real numbers a, b, c,

$$(a^2+2)(b^2+2)(c^2+2) \geq 9(ab+bc+ca).$$

First Solution. Choose $A, B, C \in \left(0, \frac{\pi}{2}\right)$ with $a = \sqrt{2}\tan A$, $b = \sqrt{2}\tan B$, and $c = \sqrt{2}\tan C$. Using the trigonometric identity $1+\tan^2\theta = \frac{1}{cos^2\theta}$, one may rewrite it as

$$\frac{4}{9} \geq \cos A\cos B\cos C\left(\cos A\sin B\sin C + \sin A\cos B\sin C + \sin A\sin B\cos C\right).$$

One may easily check the following trigonometric identity

$$\cos(A+B+C) = \cos A\cos B\cos C - \cos A\sin B\sin C - \sin A\cos B\sin C - \sin A\sin B\cos C.$$

Then, the above trigonometric inequality takes the form

$$\frac{4}{9} \geq \cos A\cos B\cos C\left(\cos A\cos B\cos C - \cos(A+B+C)\right).$$

Let $\theta = \frac{A+B+C}{3}$. Applying The AM-GM Inequality and Jesen's Inequality, we have

$$\cos A\cos B\cos C \leq \left(\frac{\cos A+\cos B+\cos C}{3}\right)^3 \leq \cos^3\theta.$$

We now need to show that

$$\frac{4}{9} \geq \cos^3\theta(\cos^3\theta - \cos 3\theta).$$

Using the trigonometric identity

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta \quad \text{or} \quad \cos^3\theta - \cos 3\theta = 3\cos\theta - 3\cos^3\theta,$$

it becomes

$$\frac{4}{27} \geq \cos^4\theta\left(1-\cos^2\theta\right),$$

which follows from The AM-GM Inequality

$$\left(\frac{\cos^2\theta}{2}\cdot\frac{\cos^2\theta}{2}\cdot\left(1-\cos^2\theta\right)\right)^{\frac{1}{3}} \leq \frac{1}{3}\left(\frac{\cos^2\theta}{2}+\frac{\cos^2\theta}{2}+\left(1-\cos^2\theta\right)\right) = \frac{1}{3}.$$

One find that the equality holds if and only if $\tan A = \tan B = \tan C = \frac{1}{\sqrt{2}}$ if and only if $a = b = c = 1$. $\square$

Epsilon 56. (Latvia 2002) Let a, b, c, d be the positive real numbers such that

$$\frac{1}{1+a^4}+\frac{1}{1+b^4}+\frac{1}{1+c^4}+\frac{1}{1+d^4}=1.$$

Prove that $abcd \geq 3$.

First Solution. We can write $a^2 = \tan A$, $b^2 = \tan B$, $c^2 = \tan C$, $d^2 = \tan D$, where $A, B, C, D \in \left(0, \frac{\pi}{2}\right)$. Then, the algebraic identity becomes the following trigonometric identity :

$$\cos^2 A + \cos^2 B + \cos^2 C + \cos^2 D = 1.$$

Applying The AM-GM Inequality, we obtain

$$\sin^2 A = 1 - \cos^2 A = \cos^2 B + \cos^2 C + \cos^2 D \geq 3 (\cos B \cos C \cos D)^{\frac{2}{3}}.$$

Similarly, we obtain

$$\sin^2 B \geq 3 (\cos C \cos D \cos A)^{\frac{2}{3}}, \sin^2 C \geq 3 (\cos D \cos A \cos B)^{\frac{2}{3}}, \text{ and } \sin^2 D \geq 3 (\cos A \cos B \cos C)^{\frac{2}{3}}.$$

Multiplying these four inequalities, we get the result! $\square$

Epsilon 57. (Korea 1998) Let x, y, z be the positive reals with $x + y + z = xyz$. Show that

$$\frac{1}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+y^2}} + \frac{1}{\sqrt{1+z^2}} \le \frac{3}{2}.$$

First Solution. We give a convexity proof. We can write $x = \tan A$, $y = \tan B$, $z = \tan C$, where $A, B, C \in \left(0, \frac{\pi}{2}\right)$. Using the fact that $1 + \tan^2\theta = \left(\frac{1}{\cos\theta}\right)^2$, we rewrite it in the terms of A, B, C :

$$\cos A + \cos B + \cos C \le \frac{3}{2}.$$

It follows from $\tan(\pi - C) = -z = \frac{x+y}{1-xy} = \tan(A + B)$ and from $\pi - C, A + B \in (0, \pi)$ that $\pi - C = A + B$ or $A + B + C = \pi$. Hence, it suffices to show the following. ☐

Epsilon 58. (USA 2001) Let a, b, and c be nonnegative real numbers such that $a^2 + b^2 + c^2 + abc = 4$. Prove that $0 \le ab + bc + ca - abc \le 2$.

Solution. Notice that $a, b, c > 1$ implies that $a^2 + b^2 + c^2 + abc > 4$. If $a \le 1$, then we have $ab + bc + ca - abc \ge (1 - a)bc \ge 0$. We now prove that $ab + bc + ca - abc \le 2$. Letting $a = 2p$, $b = 2q$, $c = 2r$, we get $p^2 + q^2 + r^2 + 2pqr = 1$. By the above exercise, we can write

$$a = 2\cos A,\ b = 2\cos B,\ c = 2\cos C \ \text{ for some } A, B, C \in \left[0, \frac{\pi}{2}\right] \text{ with } A + B + C = \pi.$$

We are required to prove

$$\cos A \cos B + \cos B \cos C + \cos C \cos A - 2\cos A \cos B \cos C \le \frac{1}{2}.$$

One may assume that $A \ge \frac{\pi}{3}$ or $1 - 2\cos A \ge 0$. Note that

$$\cos A \cos B + \cos B \cos C + \cos C \cos A - 2\cos A \cos B \cos C = \cos A(\cos B + \cos C) + \cos B \cos C(1 - 2\cos A).$$

We apply Jensen's Inequality to deduce $\cos B + \cos C \le \frac{3}{2} - \cos A$. Note that $2\cos B \cos C = \cos(B - C) + \cos(B + C) \le 1 - \cos A$. These imply that

$$\cos A(\cos B + \cos C) + \cos B \cos C(1 - 2\cos A) \le \cos A\left(\frac{3}{2} - \cos A\right) + \left(\frac{1 - \cos A}{2}\right)(1 - 2\cos A).$$

However, it's easy to verify that $\cos A\left(\frac{3}{2} - \cos A\right) + \left(\frac{1-\cos A}{2}\right)(1 - 2\cos A) = \frac{1}{2}$. $\square$

Epsilon 59. [IMO 2001/2 KOR] *Let a, b, c be positive real numbers. Prove that*

$$\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ca}}+\frac{c}{\sqrt{c^2+8ab}}\geq 1.$$

First Solution. To remove the square roots, we make the following substitution :

$$x=\frac{a}{\sqrt{a^2+8bc}},\ \ y=\frac{b}{\sqrt{b^2+8ca}},\ \ z=\frac{c}{\sqrt{c^2+8ab}}.$$

Clearly, $x, y, z \in (0,1)$. Our aim is to show that $x+y+z \geq 1$. We notice that

$$\frac{a^2}{8bc}=\frac{x^2}{1-x^2},\ \ \frac{b^2}{8ac}=\frac{y^2}{1-y^2},\ \ \frac{c^2}{8ab}=\frac{z^2}{1-z^2}\ \ \Longrightarrow\ \ \frac{1}{512}=\left(\frac{x^2}{1-x^2}\right)\left(\frac{y^2}{1-y^2}\right)\left(\frac{z^2}{1-z^2}\right).$$

Hence, we need to show that

$$x+y+z\geq 1,\ \text{where } 0<x,y,z<1 \text{ and } (1-x^2)(1-y^2)(1-z^2)=512(xyz)^2.$$

However, $1 > x+y+z$ implies that, by The AM-GM Inequality,

$(1-x^2)(1-y^2)(1-z^2) > ((x+y+z)^2-x^2)((x+y+z)^2-y^2)((x+y+z)^2-z^2) = (x+x+y+z)(y+z)$
$(x+y+y+z)(z+x)(x+y+z+z)(x+y) \geq 4(x^2yz)^{\frac{1}{4}}\cdot 2(yz)^{\frac{1}{2}}\ \cdot 4(y^2zx)^{\frac{1}{4}}\cdot 2(zx)^{\frac{1}{2}}\ \cdot 4(z^2xy)^{\frac{1}{4}}\cdot 2(xy)^{\frac{1}{2}}$
$= 512(xyz)^2$. This is a contradiction ! $\square$

Epsilon 60. [IMO 1995/2 RUS] *Let a, b, c be positive numbers such that $abc = 1$. Prove that*
$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

Second Solution. After the substitution $a = \frac{1}{x}$, $b = \frac{1}{y}$, $c = \frac{1}{z}$, we get $xyz = 1$. The inequality takes the form
$$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} \geq \frac{3}{2}.$$
It follows from The Cauchy-Schwarz Inequality that
$$[(y+z)+(z+x)+(x+y)]\left(\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y}\right) \geq (x+y+z)^2$$
so that, by The AM-GM Inequality,
$$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} \geq \frac{x+y+z}{2} \geq \frac{3(xyz)^{\frac{1}{3}}}{2} = \frac{3}{2}.$$
$\square$

Epsilon 61. (Korea 1998) Let x, y, z be the positive reals with $x+y+z=xyz$. Show that

$$\frac{1}{\sqrt{1+x^2}}+\frac{1}{\sqrt{1+y^2}}+\frac{1}{\sqrt{1+z^2}}\leq\frac{3}{2}.$$

Second Solution. The starting point is letting $a=\frac{1}{x}$, $b=\frac{1}{y}$, $c=\frac{1}{z}$. We find that $a+b+c=abc$ is equivalent to $1=xy+yz+zx$. The inequality becomes

$$\frac{x}{\sqrt{x^2+1}}+\frac{y}{\sqrt{y^2+1}}+\frac{z}{\sqrt{z^2+1}}\leq\frac{3}{2}$$

or

$$\frac{x}{\sqrt{x^2+xy+yz+zx}}+\frac{y}{\sqrt{y^2+xy+yz+zx}}+\frac{z}{\sqrt{z^2+xy+yz+zx}}\leq\frac{3}{2}$$

or

$$\frac{x}{\sqrt{(x+y)(x+z)}}+\frac{y}{\sqrt{(y+z)(y+x)}}+\frac{z}{\sqrt{(z+x)(z+y)}}\leq\frac{3}{2}.$$

By the AM-GM inequality, we have

$$\frac{x}{\sqrt{(x+y)(x+z)}}=\frac{x\sqrt{(x+y)(x+z)}}{(x+y)(x+z)}\leq\frac{1}{2}\frac{x[(x+y)+(x+z)]}{(x+y)(x+z)}=\frac{1}{2}\left(\frac{x}{x+z}+\frac{x}{x+z}\right).$$

In a like manner, we obtain

$$\frac{y}{\sqrt{(y+z)(y+x)}}\leq\frac{1}{2}\left(\frac{y}{y+z}+\frac{y}{y+x}\right)\text{ and }\frac{z}{\sqrt{(z+x)(z+y)}}\leq\frac{1}{2}\left(\frac{z}{z+x}+\frac{z}{z+y}\right).$$

Adding these three yields the required result. $\square$

Epsilon 62. [IMO 2000/2 USA] *Let a, b, c be positive numbers such that $abc = 1$. Prove that*

$$\left(a-1+\frac{1}{b}\right)\left(b-1+\frac{1}{c}\right)\left(c-1+\frac{1}{a}\right) \le 1.$$

Second Solution. ([IV], Ilan Vardi) Since $abc = 1$, we may assume that $a \ge 1 \ge b$. [17] It follows that

$$1-\left(a-1+\frac{1}{b}\right)\left(b-1+\frac{1}{c}\right)\left(c-1+\frac{1}{a}\right) = \left(c+\frac{1}{c}-2\right)\left(a+\frac{1}{b}-1\right)+\frac{(a-1)(1-b)}{a}.$$ [18]

□

Third Solution. As in the first solution, after the substitution $a = \frac{x}{y}$, $b = \frac{y}{z}$, $c = \frac{z}{x}$ for x, y, $z > 0$, we can rewrite it as $xyz \ge (y+z-x)(z+x-y)(x+y-z)$. Without loss of generality, we can assume that $z \ge y \ge x$. Set $y - x = p$ and $z - x = q$ with $p, q \ge 0$. It's straightforward to verify that

$$xyz-(y+z-x)(z+x-y)(x+y-z) = (p^2-pq+q^2)x+(p^3+q^3-p^2q-pq^2).$$

Since $p^2 - pq + q^2 \ge (p-q)^2 \ge 0$ and $p^3 + q^3 - p^2q - pq^2 = (p-q)^2(p+q) \ge 0$, we get the result. □

Fourth Solution. (From the IMO 2000 Short List) Using the condition $abc = 1$, it's straightforward to verify the equalities

$$2 = \frac{1}{a}\left(a-1+\frac{1}{b}\right)+c\left(b-1+\frac{1}{c}\right),$$
$$2 = \frac{1}{b}\left(b-1+\frac{1}{c}\right)+a\left(c-1+\frac{1}{a}\right),$$
$$2 = \frac{1}{c}\left(c-1+\frac{1}{a}\right)+b\left(a-1+\frac{1}{c}\right).$$

In particular, they show that at most one of the numbers $u = a-1+\frac{1}{b}$, $v = b-1+\frac{1}{c}$, $w = c-1+\frac{1}{a}$ is negative. If there is such a number, we have

$$\left(a-1+\frac{1}{b}\right)\left(b-1+\frac{1}{c}\right)\left(c-1+\frac{1}{a}\right) = uvw < 0 < 1.$$

And if $u, v, w \ge 0$, The AM-GM Inequality yields

$$2 = \frac{1}{a}u+cv \ge 2\sqrt{\frac{c}{a}uv},\quad 2 = \frac{1}{b}v+aw \ge 2\sqrt{\frac{a}{b}vw},\quad 2 = \frac{1}{c}w+aw \ge 2\sqrt{\frac{b}{c}wu}.$$

Thus, $uv \le \frac{a}{c}$, $vw \le \frac{b}{a}$, $wu \le \frac{c}{b}$, so $(uvw)^2 \le \frac{a}{c}\cdot\frac{b}{a}\cdot\frac{c}{b} = 1$. Since $u, v, w \ge 0$, this completes the proof. □

[17] Why? Note that the inequality is not symmetric in the three variables. Check it!
[18] For a verification of the identity, see [IV].

Epsilon 63. *Let a, b, c be positive real numbers satisfying $a+b+c=1$. Show that*

$$\frac{a}{a+bc}+\frac{b}{b+ca}+\frac{\sqrt{abc}}{c+ab}\leq 1+\frac{3\sqrt{3}}{4}.$$

Solution. We want to establish that

$$\frac{1}{1+\frac{bc}{a}}+\frac{1}{1+\frac{ca}{b}}+\frac{\sqrt{\frac{ab}{c}}}{1+\frac{ab}{c}}\leq 1+\frac{3\sqrt{3}}{4}.$$

Set $x=\sqrt{\frac{bc}{a}}$, $y=\sqrt{\frac{ca}{b}}$, $z=\sqrt{\frac{ab}{c}}$. We need to prove that

$$\frac{1}{1+x^2}+\frac{1}{1+y^2}+\frac{z}{1+z^2}\leq 1+\frac{3\sqrt{3}}{4},$$

where $x,y,z>0$ and $xy+yz+zx=1$. It's not hard to show that there exists $A,B,C\in(0,\pi)$ with

$$x=\tan\frac{A}{2}, y=\tan\frac{B}{2}, z=\tan\frac{C}{2}, \text{ and } A+B+C=\pi.$$

The inequality becomes

$$\frac{1}{1+\left(\tan\frac{A}{2}\right)^2}+\frac{1}{1+\left(\tan\frac{B}{2}\right)^2}+\frac{\tan\frac{C}{2}}{1+\left(\tan\frac{C}{2}\right)^2}\leq 1+\frac{3\sqrt{3}}{4}$$

or

$$1+\frac{1}{2}\left(\cos A+\cos B+\sin C\right)\leq 1+\frac{3\sqrt{3}}{4}$$

or

$$\cos A+\cos B+\sin C\leq\frac{3\sqrt{3}}{2}.$$

□

Note that $\cos A+\cos B=2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$. Since $\left|\frac{A-B}{2}\right|<\frac{\pi}{2}$, this means that

$$\cos A+\cos B\leq 2\cos\left(\frac{A+B}{2}\right)=2\cos\left(\frac{\pi-C}{2}\right).$$

It will be enough to show that

$$2\cos\left(\frac{\pi-C}{2}\right)+\sin C\leq\frac{3\sqrt{3}}{2},$$

where $C\in(0,\pi)$. This is a one-variable inequality.[19] It's left as an exercise for the reader.

[19] *Differentiate!* Shiing-Shen Chern

Epsilon 64. (Latvia 2002) Let a, b, c, d be the positive real numbers such that

$$\frac{1}{1+a^4}+\frac{1}{1+b^4}+\frac{1}{1+c^4}+\frac{1}{1+d^4}=1.$$

Prove that $abcd \geq 3$.

Second Solution. (given by Jeong Soo Sim at the KMO Weekend Program 2007) We need to prove the inequality $a^4b^4c^4d^4 \geq 81$. After making the substitution

$$A=\frac{1}{1+a^4},\ B=\frac{1}{1+b^4},\ C=\frac{1}{1+c^4},\ D=\frac{1}{1+d^4},$$

we obtain

$$a^4=\frac{1-A}{A},\ b^4=\frac{1-B}{B},\ c^4=\frac{1-C}{C},\ d^4=\frac{1-D}{D}.$$

The constraint becomes $A+B+C+D=1$ and the inequality can be written as

$$\frac{1-A}{A}\cdot\frac{1-B}{B}\cdot\frac{1-C}{C}\cdot\frac{1-D}{D}\geq 81.$$

or

$$\frac{B+C+D}{A}\cdot\frac{C+D+A}{B}\cdot\frac{D+A+B}{C}\cdot\frac{A+B+C}{D}\geq 81.$$

or

$$(B+C+D)(C+D+A)(D+A+B)(A+B+C)\geq 81ABCD.$$

However, this is an immediate consequence of The AM-GM Inequality:

$$(B+C+D)(C+D+A)(D+A+B)(A+B+C)\geq 3\,(BCD)^{\frac{1}{3}}\cdot 3\,(CDA)^{\frac{1}{3}}\cdot 3\,(DAB)^{\frac{1}{3}}\cdot 3\,(ABC)^{\frac{1}{3}}\,.$$

$\square$

Epsilon 65. [LL 1992 UNK] (Iran 1998) Prove that, for all $x, y, z > 1$ such that $\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = 2$,

$$\sqrt{x+y+z} \geq \sqrt{x-1}+\sqrt{y-1}+\sqrt{z-1}.$$

First Solution. We begin with the algebraic substitution $a = \sqrt{x-1}$, $b = \sqrt{y-1}$, $c = \sqrt{z-1}$. Then, the condition becomes

$$\frac{1}{1+a^2}+\frac{1}{1+b^2}+\frac{1}{1+c^2} = 2 \quad \Leftrightarrow \quad a^2b^2+b^2c^2+c^2a^2+2a^2b^2c^2 = 1$$

and the inequality is equivalent to

$$\sqrt{a^2+b^2+c^2+3} \geq a+b+c \quad \Leftrightarrow \quad ab+bc+ca \leq \frac{3}{2}.$$

Let $p = bc$, $q = ca$, $r = ab$. Our job is to prove that $p+q+r \leq \frac{3}{2}$ where $p^2+q^2+r^2+2pqr = 1$. Now, we can make the trigonometric substitution

$$p = \cos A,\ q = \cos B,\ r = \cos C \ \text{ for some } A, B, C \in \left(0, \frac{\pi}{2}\right) \text{ with } A+B+C = \pi.$$

What we need to show is now that $\cos A+\cos B+\cos C \leq \frac{3}{2}$. It follows from Jensen's Inequality. $\square$

Epsilon 66. (Belarus 1998) Prove that, for all $a, b, c > 0$,

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{a+b}{b+c} + \frac{b+c}{c+a} + 1.$$

Solution. After writing $x = \frac{a}{b}$ and $y = \frac{c}{b}$, we get

$$\frac{c}{a} = \frac{y}{x}, \quad \frac{a+b}{b+c} = \frac{x+1}{1+y}, \quad \frac{b+c}{c+a} = \frac{1+y}{y+x}.$$

One may rewrite the inequality as

$$x^3y^2 + x^2 + x + y^3 + y^2 \geq x^2y + 2xy + 2xy^2.$$

Apply The AM-GM Inequality to obtain

$$\frac{x^3y^2 + x}{2} \geq x^2y, \quad \frac{x^3y^2 + x + y^3 + y^3}{2} \geq 2xy^2, \quad x^2 + y^2 \geq 2xy.$$

Adding these three inequalities, we get the result. The equality holds if and only if $x = y = 1$ or $a = b = c$. $\square$

Epsilon 67. [SL 2001] *Let $x_1, \cdots, x_n$ be arbitrary real numbers. Prove the inequality.*

$$\frac{x_1}{1+{x_1}^2}+\frac{x_2}{1+{x_1}^2+{x_2}^2}+\cdots+\frac{x_n}{1+{x_1}^2+\cdots+{x_n}^2}<\sqrt{n}.$$

First Solution. We only consider the case when $x_1, \cdots, x_n$ are all nonnegative real numbers.(Why?)[20] Let $x_0 = 1$. After the substitution $y_i = {x_0}^2 + \cdots + {x_i}^2$ for all $i = 0, \cdots, n$, we obtain $x_i = \sqrt{y_i - y_{i-1}}$. We need to prove the following inequality

$$\sum_{i=0}^{n} \frac{\sqrt{y_i - y_{i-1}}}{y_i} < \sqrt{n}.$$

Since $y_i \geq y_{i-1}$ for all $i = 1, \cdots, n$, we have an upper bound of the left hand side:

$$\sum_{i=0}^{n} \frac{\sqrt{y_i - y_{i-1}}}{y_i} \leq \sum_{i=0}^{n} \frac{\sqrt{y_i - y_{i-1}}}{\sqrt{y_i y_{i-1}}} = \sum_{i=0}^{n} \sqrt{\frac{1}{y_{i-1}} - \frac{1}{y_i}}$$

We now apply the Cauchy-Schwarz inequality to give an upper bound of the last term:

$$\sum_{i=0}^{n} \sqrt{\frac{1}{y_{i-1}} - \frac{1}{y_i}} \leq \sqrt{n \sum_{i=0}^{n} \left(\frac{1}{y_{i-1}} - \frac{1}{y_i}\right)} = \sqrt{n \left(\frac{1}{y_0} - \frac{1}{y_n}\right)}.$$

Since $y_0 = 1$ and $y_n > 0$, this yields the desired upper bound $\sqrt{n}$.

$\square$

Second Solution. We may assume that $x_1, \cdots, x_n$ are all nonnegative real numbers. Let $x_0 = 0$. We make the following *algebraic* substitution

$$t_i = \frac{x_i}{\sqrt{{x_0}^2 + \cdots + {x_i}^2}},\ c_i = \frac{1}{\sqrt{1+{t_i}^2}} \text{ and } s_i = \frac{t_i}{\sqrt{1+{t_i}^2}}$$

for all $i = 0, \cdots, n$. It's an easy exercise to show that $\frac{x_i}{{x_0}^2+\cdots+{x_i}^2} = c_0 \cdots c_i s_i$. Since $s_i = \sqrt{1 - {c_i}^2}$, the desired inequality becomes

$$c_0 c_1 \sqrt{1-{c_1}^2} + c_0 c_1 c_2 \sqrt{1-{c_2}^2} + \cdots + c_0 c_1 \cdots c_n \sqrt{1-{c_n}^2} < \sqrt{n}.$$

Since $0 < c_i \leq 1$ for all $i = 1, \cdots, n$, we have

$$\sum_{i=1}^{n} c_0 \cdots c_i \sqrt{1-{c_i}^2} \leq \sum_{i=1}^{n} c_0 \cdots c_{i-1} \sqrt{1-{c_i}^2} = \sum_{i=1}^{n} \sqrt{(c_0 \cdots c_{i-1})^2 - (c_0 \cdots c_{i-1} c_i)^2}.$$

Since $c_0 = 1$, by The Cauchy-Schwarz Inequality, we obtain

$$\sum_{i=1}^{n} \sqrt{(c_0 \cdots c_{i-1})^2 - (c_0 \cdots c_{i-1} c_i)^2} \leq \sqrt{n \sum_{i=1}^{n} [(c_0 \cdots c_{i-1})^2 - (c_0 \cdots c_{i-1} c_i)^2]} = \sqrt{n\,[1-(c_0 \cdots c_n)^2]}.$$

$\square$

[20] $\frac{x_1}{1+{x_1}^2}+\frac{x_2}{1+{x_1}^2+{x_2}^2}+\cdots+\frac{x_n}{1+{x_1}^2+\cdots+{x_n}^2} \leq \frac{|x_1|}{1+{x_1}^2}+\frac{|x_2|}{1+{x_1}^2+{x_2}^2}+\cdots+\frac{|x_n|}{1+{x_1}^2+\cdots+{x_n}^2}.$

Epsilon 68. *Let a, b, c be the lengths of a triangle. Show that*

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}<2.$$

Solution. We don't employ The Ravi Substitution! It follows from the triangle inequality that

$$\sum_{\text{cyclic}}\frac{a}{b+c}<\sum_{\text{cyclic}}\frac{a}{\frac{1}{2}(a+b+c)}=2.$$

$\square$

Epsilon 69. [IMO 2001/2 KOR] *Let a, b, c be positive real numbers. Prove that*

$$\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ca}}+\frac{c}{\sqrt{c^2+8ab}}\geq 1.$$

Second Solution. Let's try to find a new lower bound of $(x+y+z)^2$ where $x, y, z > 0$. There are well-known lower bounds such as $3(xy+yz+zx)$ and $9(xyz)^{\frac{2}{3}}$. Here, we break the symmetry. We notice that

$$(x+y+z)^2 = x^2+y^2+z^2+xy+xy+yz+yz+zx+zx.$$

We apply The AM-GM Inequality to the right hand side except the term x^2:

$$y^2+z^2+xy+xy+yz+yz+zx+zx \geq 8x^{\frac{1}{2}}y^{\frac{3}{4}}z^{\frac{3}{4}}\ .$$

It follows that

$$(x+y+z)^2 \geq x^2+8x^{\frac{1}{2}}y^{\frac{3}{4}}z^{\frac{3}{4}} = x^{\frac{1}{2}}\left(x^{\frac{3}{2}}+8y^{\frac{3}{4}}z^{\frac{3}{4}}\right).$$

We proved the estimation, for $x, y, z > 0$,

$$x+y+z \geq \sqrt{x^{\frac{1}{2}}\left(x^{\frac{3}{2}}+8y^{\frac{3}{4}}z^{\frac{3}{4}}\right)}.$$

It follows that

$$\sum_{\text{cyclic}}\frac{x^{\frac{3}{4}}}{\sqrt{x^{\frac{3}{2}}+8y^{\frac{3}{4}}z^{\frac{3}{4}}}} \geq \sum_{\text{cyclic}}\frac{x}{x+y+z} = 1.$$

After the substitution $x = a^{\frac{4}{3}}, y = b^{\frac{4}{3}}$, and $z = c^{\frac{4}{3}}$, it now becomes the inequality

$$\sum_{\text{cyclic}}\frac{a}{\sqrt{a^2+8bc}} \geq 1.$$

□

Epsilon 70. [IMO 2005/3 KOR] *Let x, y, and z be positive numbers such that $xyz \geq 1$. Prove that*

$$\frac{x^5-x^2}{x^5+y^2+z^2}+\frac{y^5-y^2}{y^5+z^2+x^2}+\frac{z^5-z^2}{z^5+x^2+y^2}\geq 0.$$

First Solution. It's equivalent to the following inequality

$$\left(\frac{x^2-x^5}{x^5+y^2+z^2}+1\right)+\left(\frac{y^2-y^5}{y^5+z^2+x^2}+1\right)+\left(\frac{z^2-z^5}{z^5+x^2+y^2}+1\right)\leq 3$$

or

$$\frac{x^2+y^2+z^2}{x^5+y^2+z^2}+\frac{x^2+y^2+z^2}{y^5+z^2+x^2}+\frac{x^2+y^2+z^2}{z^5+x^2+y^2}\leq 3.$$

With The Cauchy-Schwarz Inequality and the fact that $xyz \geq 1$, we have

$$(x^5+y^2+z^2)(yz+y^2+z^2)\geq (x^2+y^2+z^2)^2$$

or

$$\frac{x^2+y^2+z^2}{x^5+y^2+z^2}\leq \frac{yz+y^2+z^2}{x^2+y^2+z^2}.$$

Taking the cyclic sum, we reach

$$\frac{x^2+y^2+z^2}{x^5+y^2+z^2}+\frac{x^2+y^2+z^2}{y^5+z^2+x^2}+\frac{x^2+y^2+z^2}{z^5+x^2+y^2}\leq 2+\frac{xy+yz+zx}{x^2+y^2+z^2}\leq 3.$$

□

Second Solution. The main idea is to think of 1 as follows :

$$\frac{x^5}{x^5+y^2+z^2}+\frac{y^5}{y^5+z^2+x^2}+\frac{z^5}{z^5+x^2+y^2}\geq 1\geq \frac{x^2}{x^5+y^2+z^2}+\frac{y^2}{y^5+z^2+x^2}+\frac{z^2}{z^5+x^2+y^2}.$$

We first show the left-hand. It follows from $y^4+z^4 \geq y^3z+yz^3 = yz(y^2+z^2)$ that

$$x(y^4+z^4)\geq xyz(y^2+z^2)\geq y^2+z^2 \ \text{ or } \ \frac{x^5}{x^5+y^2+z^2}\geq \frac{x^5}{x^5+xy^4+xz^4}=\frac{x^4}{x^4+y^4+z^4}.$$

Taking the cyclic sum, we have the required inequality. It remains to show the right-hand. As in the first solution, The Cauchy-Schwarz Inequality and $xyz \geq 1$ imply that

$$(x^5+y^2+z^2)(yz+y^2+z^2)\geq (x^2+y^2+z^2)^2 \ \text{ or } \ \frac{x^2(yz+y^2+z^2)}{(x^2+y^2+z^2)^2}\geq \frac{x^2}{x^5+y^2+z^2}.$$

Taking the cyclic sum, we have

$$\sum_{\text{cyclic}}\frac{x^2(yz+y^2+z^2)}{(x^2+y^2+z^2)^2}\geq \sum_{\text{cyclic}}\frac{x^2}{x^5+y^2+z^2}.$$

Our job is now to establish the following homogeneous inequality

$$1\geq \sum_{\text{cyclic}}\frac{x^2(yz+y^2+z^2)}{(x^2+y^2+z^2)^2}\Leftrightarrow (x^2+y^2+z^2)^2\geq 2\sum_{\text{cyclic}}x^2y^2+\sum_{\text{cyclic}}x^2yz \Leftrightarrow \sum_{\text{cyclic}}x^4\geq \sum_{\text{cyclic}}x^2yz.$$

However, by The AM-GM Inequality, we obtain

$$\sum_{\text{cyclic}}x^4=\sum_{\text{cyclic}}\frac{x^4+y^4}{2}\geq \sum_{\text{cyclic}}x^2y^2=\sum_{\text{cyclic}}x^2\left(\frac{y^2+z^2}{2}\right)\geq \sum_{\text{cyclic}}x^2yz.$$

□

Remark 8.2. *Here is an alternative way to reach the right hand side inequality. We claim that*

$$\frac{2x^4+y^4+z^4+4x^2y^2+4x^2z^2}{4(x^2+y^2+z^2)^2} \geq \frac{x^2}{x^5+y^2+z^2}.$$

We do this by proving

$$\frac{2x^4+y^4+z^4+4x^2y^2+4x^2z^2}{4(x^2+y^2+z^2)^2} \geq \frac{x^2yz}{x^4+y^3z+yz^3}$$

because $xyz \geq 1$ *implies that*

$$\frac{x^2yz}{x^4+y^3z+yz^3} = \frac{x^2}{\frac{x^5}{xyz}+y^2+z^2} \geq \frac{x^2}{x^5+y^2+z^2}.$$

Hence, we need to show the homogeneous inequality

$$(2x^4+y^4+z^4+4x^2y^2+4x^2z^2)(x^4+y^3z+yz^3) \geq 4x^2yz(x^2+y^2+z^2)^2.$$

However, this is a straightforward consequence of The AM-GM Inequality.

$$\begin{aligned}
& (2x^4+y^4+z^4+4x^2y^2+4x^2z^2)(x^4+y^3z+yz^3) - 4x^2yz(x^2+y^2+z^2)^2 \\
= \; & (x^8+x^4y^4+x^6y^2+x^6y^2+y^7z+y^3z^5)+(x^8+x^4z^4+x^6z^2+x^6z^2+yz^7+y^5z^3) \\
& +2(x^6y^2+x^6z^2) - 6x^4y^3z - 6x^4yz^3 - 2x^6yz \\
\geq \; & 6\sqrt[6]{x^8\cdot x^4y^4\cdot x^6y^2\cdot x^6y^2\cdot y^7z\cdot y^3z^5} + 6\sqrt[6]{x^8\cdot x^4z^4\cdot x^6z^2\cdot x^6z^2\cdot yz^7\cdot y^5z^3} \\
& +2\sqrt{x^6y^2\cdot x^6z^2} - 6x^4y^3z - 6x^4yz^3 - 2x^6yz \\
= \; & 0.
\end{aligned}$$

Taking the cyclic sum, we obtain

$$1 = \sum_{\text{cyclic}} \frac{2x^4+y^4+z^4+4x^2y^2+4x^2z^2}{4(x^2+y^2+z^2)^2} \geq \sum_{\text{cyclic}} \frac{x^2}{x^5+y^2+z^2}.$$

Third Solution. (by an IMO 2005 contestant Iurie Boreico[21] from Moldova) We establish that

$$\frac{x^5-x^2}{x^5+y^2+z^2} \geq \frac{x^5-x^2}{x^3(x^2+y^2+z^2)}.$$

It follows immediately from the identity

$$\frac{x^5-x^2}{x^5+y^2+z^2} - \frac{x^5-x^2}{x^3(x^2+y^2+z^2)} = \frac{(x^3-1)^2x^2(y^2+z^2)}{x^3(x^2+y^2+z^2)(x^5+y^2+z^2)}.$$

Taking the cyclic sum and using $xyz \geq 1$, we have

$$\sum_{\text{cyclic}} \frac{x^5-x^2}{x^5+y^2+z^2} \geq \frac{1}{x^5+y^2+z^2}\sum_{\text{cyclic}}\left(x^2-\frac{1}{x}\right) \geq \frac{1}{x^5+y^2+z^2}\sum_{\text{cyclic}}\left(x^2-yz\right) \geq 0.$$

$\square$

[21]He received the special prize for this solution.

Epsilon 71. (KMO Weekend Program 2007) Prove that, for all $a, b, c, x, y, z > 0$,

$$\frac{ax}{a+x} + \frac{by}{b+y} + \frac{cz}{c+z} \leq \frac{(a+b+c)(x+y+z)}{a+b+c+x+y+z}.$$

Solution. (by Sanghoon at the KMO Weekend Program 2007) We need the following lemma:

Lemma 8.2. *For all $p, q, \omega_1, \omega_2 > 0$, we have*

$$\frac{pq}{p+q} \leq \frac{{\omega_1}^2 p + {\omega_2}^2 q}{(\omega_1 + \omega_2)^2}.$$

Proof. After expanding, it becomes

$$(p+q)\left({\omega_1}^2 p + {\omega_2}^2 q\right) - (\omega_1 + \omega_2)^2 pq \geq 0.$$

However, it can be written as

$$(\omega_1 p - \omega_2 q)^2 \geq 0.$$

□

Now, taking $(p, q, \omega_1, \omega_2) = (a, x, x+y+z, a+b+c)$ in the lemma, we get

$$\frac{ax}{a+x} \leq \frac{(x+y+z)^2 a + (a+b+c)^2 x}{(x+y+z+a+b+c)^2}.$$

Similarly, we obtain

$$\frac{by}{b+y} \leq \frac{(x+y+z)^2 b + (a+b+c)^2 y}{(x+y+z+a+b+c)^2}$$

and

$$\frac{cz}{c+z} \leq \frac{(x+y+z)^2 c + (a+b+c)^2 z}{(x+y+z+a+b+c)^2}.$$

Adding the above three inequalities, we get

$$\frac{ax}{a+x} + \frac{by}{b+y} + \frac{cz}{c+z} \leq \frac{(x+y+z)^2(a+b+c) + (a+b+c)^2(x+y+z)}{(x+y+z+a+b+c)^2}.$$

or

$$\frac{ax}{a+x} + \frac{by}{b+y} + \frac{cz}{c+z} \leq \frac{(a+b+c)(x+y+z)}{a+b+c+x+y+z},$$

as desired.

□

Epsilon 72. (USAMO Summer Program 2002) Let a, b, c be positive real numbers. Prove that

$$\left(\frac{2a}{b+c}\right)^{\frac{2}{3}}+\left(\frac{2b}{c+a}\right)^{\frac{2}{3}}+\left(\frac{2c}{a+b}\right)^{\frac{2}{3}}\geq 3.$$

Proof. Establish the inequality

$$\left(\frac{2a}{b+c}\right)^{\frac{2}{3}}\geq 3\left(\frac{a}{a+b+c}\right).$$

□

Epsilon 73. (APMO 2005) Let a, b, c be positive real numbers with $abc = 8$. Prove that

$$\frac{a^2}{\sqrt{(1+a^3)(1+b^3)}} + \frac{b^2}{\sqrt{(1+b^3)(1+c^3)}} + \frac{c^2}{\sqrt{(1+c^3)(1+a^3)}} \geq \frac{4}{3}$$

Proof. Use the auxiliary inequality

$$\frac{1}{\sqrt{1+x^3}} \geq \frac{2}{2+x^2}.$$

□

Epsilon 74. (Titu Andreescu, Gabriel Dospinescu) Let x, y, and z be real numbers such that $x, y, z \leq 1$ and $x + y + z = 1$. Prove that

$$\frac{1}{1+x^2} + \frac{1}{1+y^2} + \frac{1}{1+z^2} \leq \frac{27}{10}.$$

Solution. Employ the following inequality

$$\frac{1}{1+t^2} \leq -\frac{27}{50}\left(t-2\right),$$

where $t \leq 1$. $\square$

Epsilon 75. (Japan 1997) Let a, b, and c be positive real numbers. Prove that

$$\frac{(b+c-a)^2}{(b+c)^2+a^2}+\frac{(c+a-b)^2}{(c+a)^2+b^2}+\frac{(a+b-c)^2}{(a+b)^2+c^2}\geq\frac{3}{5}.$$

Solution. Because of the homogeneity of the inequality, we may normalize to $a+b+c=1$. It takes the form

$$\frac{(1-2a)^2}{(1-a)^2+a^2}+\frac{(1-2b)^2}{(1-b)^2+b^2}+\frac{(1-2c)^2}{(1-c)^2+c^2}\geq\frac{3}{5}$$

or

$$\frac{1}{2a^2-2a+1}+\frac{1}{2b^2-2b+1}+\frac{1}{2c^2-2c+1}\leq\frac{27}{5}.$$

We find that the equation of the tangent line of $f(x)=\frac{1}{2x^2-2x+1}$ at $x=\frac{1}{3}$ is given by

$$y=\frac{54}{25}x+\frac{27}{25}$$

and that

$$f(x)-\left(\frac{54}{25}x+\frac{27}{25}\right)=-\frac{2(3x-1)^2(6x+1)}{25(2x^2-2x+1)}\leq 0.$$

for all $x>0$. It follows that

$$\sum_{\text{cyclic}}f(a)\leq\sum_{\text{cyclic}}\frac{54}{25}a+\frac{27}{25}=\frac{27}{5}.$$

□

Epsilon 76. [IMO 1984/1 FRG] *Let x, y, z be nonnegative real numbers such that $x+y+z = 1$. Prove that $0 \leq xy + yz + zx - 2xyz \leq \frac{7}{27}$.*

First Solution. Using the constraint $x+y+z = 1$, we reduce the inequality to homogeneous one:

$$0 \leq (xy + yz + zx)(x + y + z) - 2xyz \leq \frac{7}{27}(x + y + z)^3.$$

The left hand side inequality is trivial because it's equivalent to

$$0 \leq xyz + \sum_{\text{sym}} x^2y.$$

The right hand side inequality simplifies to

$$7 \sum_{\text{cyclic}} x^3 + 15xyz - 6 \sum_{\text{sym}} x^2y \geq 0.$$

In the view of

$$7 \sum_{\text{cyclic}} x^3 + 15xyz - 6 \sum_{\text{sym}} x^2y = \left(2 \sum_{\text{cyclic}} x^3 - \sum_{\text{sym}} x^2y\right) + 5\left(3xyz + \sum_{\text{cyclic}} x^3 - \sum_{\text{sym}} x^2y\right),$$

it's enough to show that

$$2 \sum_{\text{cyclic}} x^3 \geq \sum_{\text{sym}} x^2y$$

and

$$3xyz + \sum_{\text{cyclic}} x^3 \geq \sum_{\text{sym}} x^2y.$$

The first inequality follows from

$$2 \sum_{\text{cyclic}} x^3 - \sum_{\text{sym}} x^2y = \sum_{\text{cyclic}} (x^3 + y^3) - \sum_{\text{cyclic}} (x^2y + xy^2) = \sum_{\text{cyclic}} (x^3 + y^3 - x^2y - xy^2) \geq 0.$$

The second inequality can be rewritten as

$$\sum_{\text{cyclic}} x(x - y)(x - z) \geq 0,$$

which is a particular case of Schur's Theorem. □

Epsilon 77. [LL 1992 UNK] (Iran 1998) Prove that, for all $x, y, z > 1$ such that $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=2$,

$$\sqrt{x+y+z} \geq \sqrt{x-1}+\sqrt{y-1}+\sqrt{z-1}.$$

Second Solution. After the algebraic substitution $a=\frac{1}{x}$, $b=\frac{1}{y}$, $c=\frac{1}{z}$, we are required to prove that

$$\sqrt{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}} \geq \sqrt{\frac{1-a}{a}}+\sqrt{\frac{1-b}{b}}+\sqrt{\frac{1-c}{c}},$$

where $a, b, c \in (0,1)$ and $a+b+c=2$. Using the constraint $a+b+c=2$, we obtain a homogeneous inequality

$$\sqrt{\frac{1}{2}(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)} \geq \sqrt{\frac{\frac{a+b+c}{2}-a}{a}}+\sqrt{\frac{\frac{a+b+c}{2}-b}{b}}+\sqrt{\frac{\frac{a+b+c}{2}-c}{c}}$$

or

$$\sqrt{(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)} \geq \sqrt{\frac{b+c-a}{a}}+\sqrt{\frac{c+a-b}{b}}+\sqrt{\frac{a+b-c}{c}},$$

which immediately follows from The Cauchy-Schwarz Inequality

$$\sqrt{[(b+c-a)+(c+a-b)+(a+b-c)]\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)} \geq \sqrt{\frac{b+c-a}{a}}+\sqrt{\frac{c+a-b}{b}}+\sqrt{\frac{a+b-c}{c}}.$$

□

Epsilon 78. *Let x, y, z be nonnegative real numbers. Then, we have*

$$3xyz + x^3 + y^3 + z^3 \geq 2\left((xy)^{\frac{3}{2}} + (yz)^{\frac{3}{2}} + (zx)^{\frac{3}{2}}\right).$$

First Solution. By Schur's Inequality and The AM-GM Inequality, we have

$$3xyz + \sum_{\text{cyclic}} x^3 \geq \sum_{\text{cyclic}} x^2y + xy^2 \geq \sum_{\text{cyclic}} 2(xy)^{\frac{3}{2}}.$$

$\square$

Epsilon 79. *Let $t \in (0,3]$. For all $a,b,c \geq 0$, we have*

$$(3-t)+t(abc)^{\frac{2}{t}}+\sum_{\text{cyclic}} a^2 \geq 2\sum_{\text{cyclic}} ab.$$

Proof. After setting $x=a^{\frac{2}{3}}$, $y=b^{\frac{2}{3}}$, $z=c^{\frac{2}{3}}$, it becomes

$$3-t+t(xyz)^{\frac{3}{t}}+\sum_{\text{cyclic}} x^3 \geq 2\sum_{\text{cyclic}} (xy)^{\frac{3}{2}}.$$

By the previous epsilon, it will be enough to show that

$$3-t+t(xyz)^{\frac{3}{t}} \geq 3xyz,$$

which is a straightforward consequence of the weighted AM-GM inequality :

$$\frac{3-t}{3}\cdot 1+\frac{t}{3}(xyz)^{\frac{3}{t}} \geq 1^{\frac{3-t}{3}}\left((xyz)^{\frac{3}{t}}\right)^{\frac{t}{3}}=3xyz.$$

One may check that the equality holds if and only if $a=b=c=1$. □

Remark 8.3. *In particular, we obtain non-homogeneous inequalities*

$$\frac{5}{2}+\frac{1}{2}(abc)^4+a^2+b^2+c^2 \geq 2(ab+bc+ca),$$
$$2+(abc)^2+a^2+b^2+c^2 \geq 2(ab+bc+ca),$$
$$1+2abc+a^2+b^2+c^2 \geq 2(ab+bc+ca).$$

Epsilon 80. (APMO 2004/5) *Prove that, for all positive real numbers a, b, c,*

$$(a^2+2)(b^2+2)(c^2+2) \geq 9(ab+bc+ca).$$

Second Solution. After expanding, it becomes

$$8+(abc)^2+2\sum_{\text{cyclic}} a^2b^2+4\sum_{\text{cyclic}} a^2 \geq 9\sum_{\text{cyclic}} ab.$$

From the inequality $(ab-1)^2+(bc-1)^2+(ca-1)^2 \geq 0$, we obtain

$$6+2\sum_{\text{cyclic}} a^2b^2 \geq 4\sum_{\text{cyclic}} ab.$$

Hence, it will be enough to show that

$$2+(abc)^2+4\sum_{\text{cyclic}} a^2 \geq 5\sum_{\text{cyclic}} ab.$$

Since $3(a^2+b^2+c^2) \geq 3(ab+bc+ca)$, it will be enough to show that

$$2+(abc)^2+\sum_{\text{cyclic}} a^2 \geq 2\sum_{\text{cyclic}} ab,$$

which is a particular case of the previous epsilon. $\square$

Epsilon 81. [IMO 2000/2 USA] *Let a, b, c be positive numbers such that $abc = 1$. Prove that*

$$\left(a-1+\frac{1}{b}\right)\left(b-1+\frac{1}{c}\right)\left(c-1+\frac{1}{a}\right)\leq 1.$$

Second Solution. It is equivalent to the following homogeneous inequality:

$$\left(a-(abc)^{1/3}+\frac{(abc)^{2/3}}{b}\right)\left(b-(abc)^{1/3}+\frac{(abc)^{2/3}}{c}\right)\left(c-(abc)^{1/3}+\frac{(abc)^{2/3}}{a}\right)\leq abc.$$

After the substitution $a=x^3, b=y^3, c=z^3$ with $x, y, z>0$, it becomes

$$\left(x^3-xyz+\frac{(xyz)^2}{y^3}\right)\left(y^3-xyz+\frac{(xyz)^2}{z^3}\right)\left(z^3-xyz+\frac{(xyz)^2}{x^3}\right)\leq x^3y^3z^3,$$

which simplifies to

$$\left(x^2y-y^2z+z^2x\right)\left(y^2z-z^2x+x^2y\right)\left(z^2x-x^2y+y^2z\right)\leq x^3y^3z^3$$

or

$$3x^3y^3z^3+\sum_{\text{cyclic}}x^6y^3\geq\sum_{\text{cyclic}}x^4y^4z+\sum_{\text{cyclic}}x^5y^2z^2$$

or

$$3(x^2y)(y^2z)(z^2x)+\sum_{\text{cyclic}}(x^2y)^3\geq\sum_{\text{sym}}(x^2y)^2(y^2z)$$

which is a special case of Schur's Inequality. $\square$

Epsilon 82. (Tournament of Towns 1997) Let a, b, c be positive numbers such that $abc = 1$. Prove that
$$\frac{1}{a+b+1} + \frac{1}{b+c+1} + \frac{1}{c+a+1} \le 1.$$

Solution. We can rewrite the given inequality as following :
$$\frac{1}{a+b+(abc)^{1/3}} + \frac{1}{b+c+(abc)^{1/3}} + \frac{1}{c+a+(abc)^{1/3}} \le \frac{1}{(abc)^{1/3}}.$$
We make the substitution $a = x^3, b = y^3, c = z^3$ with $x, y, z > 0$. Then, it becomes
$$\frac{1}{x^3+y^3+xyz} + \frac{1}{y^3+z^3+xyz} + \frac{1}{z^3+x^3+xyz} \le \frac{1}{xyz}$$
which is equivalent to
$$xyz \sum_{\text{cyclic}} (x^3+y^3+xyz)(y^3+z^3+xyz) \le (x^3+y^3+xyz)(y^3+z^3+xyz)(z^3+x^3+xyz)$$
or
$$\sum_{\text{sym}} x^6y^3 \ge \sum_{\text{sym}} x^5y^2z^2 \quad !$$
We now obtain
$$\begin{aligned}
\sum_{\text{sym}} x^6y^3 &= \sum_{\text{cyclic}} x^6y^3 + y^6x^3 \\
&\ge \sum_{\text{cyclic}} x^5y^4 + y^5x^4 \\
&= \sum_{\text{cyclic}} x^5(y^4 + z^4) \\
&\ge \sum_{\text{cyclic}} x^5(y^2z^2 + y^2z^2) \\
&= \sum_{\text{sym}} x^5y^2z^2.
\end{aligned}$$
□

Epsilon 83. (Muirhead's Theorem) Let $a_1, a_2, a_3, b_1, b_2, b_3$ be real numbers such that $a_1 \geq a_2 \geq a_3 \geq 0, b_1 \geq b_2 \geq b_3 \geq 0, a_1 \geq b_1, a_1 + a_2 \geq b_1 + b_2, a_1 + a_2 + a_3 = b_1 + b_2 + b_3$. Let x, y, z be positive real numbers. Then, we have

$$\sum_{\text{sym}} x^{a_1} y^{a_2} z^{a_3} \geq \sum_{\text{sym}} x^{b_1} y^{b_2} z^{b_3}.$$

Solution. We distinguish two cases.

Case 1. $b_1 \geq a_2$: It follows from $a_1 \geq a_1+a_2-b_1$ and from $a_1 \geq b_1$ that $a_1 \geq max(a_1+a_2-b_1, b_1)$ so that $max(a_1, a_2) = a_1 \geq max(a_1+a_2-b_1, b_1)$. From $a_1+a_2-b_1 \geq b_1+a_3-b_1 = a_3$ and $a_1 + a_2 - b_1 \geq b_2 \geq b_3$, we have $max(a_1 + a_2 - b_1, a_3) \geq max(b_2, b_3)$. It follows that

$$\begin{aligned}
\sum_{\text{sym}} x^{a_1} y^{a_2} z^{a_3} &= \sum_{\text{cyclic}} z^{a_3}(x^{a_1} y^{a_2} + x^{a_2} y^{a_1}) \\
&\geq \sum_{\text{cyclic}} z^{a_3}(x^{a_1+a_2-b_1} y^{b_1} + x^{b_1} y^{a_1+a_2-b_1}) \\
&= \sum_{\text{cyclic}} x^{b_1}(y^{a_1+a_2-b_1} z^{a_3} + y^{a_3} z^{a_1+a_2-b_1}) \\
&\geq \sum_{\text{cyclic}} x^{b_1}(y^{b_2} z^{b_3} + y^{b_3} z^{b_2}) \\
&= \sum_{\text{sym}} x^{b_1} y^{b_2} z^{b_3}.
\end{aligned}$$

Case 2. $b_1 \leq a_2$: It follows from $3b_1 \geq b_1 + b_2 + b_3 = a_1 + a_2 + a_3 \geq b_1 + a_2 + a_3$ that $b_1 \geq a_2 + a_3 - b_1$ and that $a_1 \geq a_2 \geq b_1 \geq a_2 + a_3 - b_1$. Therefore, we have $max(a_2, a_3) \geq max(b_1, a_2 + a_3 - b_1)$ and $max(a_1, a_2 + a_3 - b_1) \geq max(b_2, b_3)$. It follows that

$$\begin{aligned}
\sum_{\text{sym}} x^{a_1} y^{a_2} z^{a_3} &= \sum_{\text{cyclic}} x^{a_1}(y^{a_2} z^{a_3} + y^{a_3} z^{a_2}) \\
&\geq \sum_{\text{cyclic}} x^{a_1}(y^{b_1} z^{a_2+a_3-b_1} + y^{a_2+a_3-b_1} z^{b_1}) \\
&= \sum_{\text{cyclic}} y^{b_1}(x^{a_1} z^{a_2+a_3-b_1} + x^{a_2+a_3-b_1} z^{a_1}) \\
&\geq \sum_{\text{cyclic}} y^{b_1}(x^{b_2} z^{b_3} + x^{b_3} z^{b_2}) \\
&= \sum_{\text{sym}} x^{b_1} y^{b_2} z^{b_3}.
\end{aligned}$$

□

Epsilon 84. [IMO 1995/2 RUS] *Let a, b, c be positive numbers such that $abc = 1$. Prove that*

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

Third Solution. It's equivalent to

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2(abc)^{4/3}}.$$

Set $a = x^3, b = y^3, c = z^3$ with $x, y, z > 0$. Then, it becomes

$$\sum_{\text{cyclic}} \frac{1}{x^9(y^3+z^3)} \geq \frac{3}{2x^4y^4z^4}.$$

Clearing denominators, this can be rewritten as

$$\sum_{\text{sym}} x^{12}y^{12} + 2\sum_{\text{sym}} x^{12}y^9z^3 + \sum_{\text{sym}} x^9y^9z^6 \geq 3\sum_{\text{sym}} x^{11}y^8z^5 + 6x^8y^8z^8$$

or

$$\left(\sum_{\text{sym}} x^{12}y^{12} - \sum_{\text{sym}} x^{11}y^8z^5\right) + 2\left(\sum_{\text{sym}} x^{12}y^9z^3 - \sum_{\text{sym}} x^{11}y^8z^5\right) + \left(\sum_{\text{sym}} x^9y^9z^6 - \sum_{\text{sym}} x^8y^8z^8\right) \geq 0,$$

By Muirhead's Theorem, every term on the left hand side is nonnegative. $\square$

Epsilon 85. (Iran 1996) Let x, y, z be positive real numbers. Prove that

$$(xy + yz + zx)\left(\frac{1}{(x+y)^2} + \frac{1}{(y+z)^2} + \frac{1}{(z+x)^2}\right) \geq \frac{9}{4}.$$

Second Solution. It's equivalent to

$$4\sum_{\text{sym}} x^5y + 2\sum_{\text{cyclic}} x^4yz + 6x^2y^2z^2 - \sum_{\text{sym}} x^4y^2 - 6\sum_{\text{cyclic}} x^3y^3 - 2\sum_{\text{sym}} x^3y^2z \geq 0.$$

We rewrite this as following

$$\left(\sum_{\text{sym}} x^5y - \sum_{\text{sym}} x^4y^2\right) + 3\left(\sum_{\text{sym}} x^5y - \sum_{\text{sym}} x^3y^3\right) + 2xyz\left(3xyz + \sum_{\text{cyclic}} x^3 - \sum_{\text{sym}} x^2y\right) \geq 0.$$

By Muirhead's Theorem and Schur's Inequality, it's a sum of three nonnegative terms. $\square$

Epsilon 86. *Let x, y, z be nonnegative real numbers with $xy + yz + zx = 1$. Prove that*

$$\frac{1}{x+y} + \frac{1}{y+z} + \frac{1}{z+x} \geq \frac{5}{2}.$$

Solution. Using $xy + yz + zx = 1$, we homogenize the given inequality as following :

$$(xy + yz + zx)\left(\frac{1}{x+y} + \frac{1}{y+z} + \frac{1}{z+x}\right)^2 \geq \left(\frac{5}{2}\right)^2$$

or

$$4\sum_{\text{sym}} x^5y + \sum_{\text{sym}} x^4yz + 14\sum_{\text{sym}} x^3y^2z + 38x^2y^2z^2 \geq \sum_{\text{sym}} x^4y^2 + 3\sum_{\text{sym}} x^3y^3$$

or

$$\left(\sum_{\text{sym}} x^5y - \sum_{\text{sym}} x^4y^2\right) + 3\left(\sum_{\text{sym}} x^5y - \sum_{\text{sym}} x^3y^3\right) + xyz\left(\sum_{\text{sym}} x^3 + 14\sum_{\text{sym}} x^2y + 38xyz\right) \geq 0.$$

By Muirhead's Theorem, we get the result. In the above inequality, without the condition $xy + yz + zx = 1$, the equality holds if and only if $x = y, z = 0$ *or* $y = z, x = 0$ *or* $z = x, y = 0$. Since $xy + yz + zx = 1$, the equality occurs when $(x, y, z) = (1, 1, 0), (1, 0, 1), (0, 1, 1)$. $\square$

Epsilon 87. [SC] If m_a,m_b,m_c are medians and r_a,r_b,r_c the exradii of a triangle, prove that

$$\frac{r_a r_b}{m_a m_b} + \frac{r_b r_c}{m_b m_c} + \frac{r_c r_a}{m_c m_a} \geq 3.$$

Solution. Set $2s = a + b + c$. Using the well-known identities

$$r_a = \sqrt{\frac{s(s-b)(s-c)}{s-a}},\ m_a = \frac{1}{2}\sqrt{2b^2+2c^2-a^2},\ etc.$$

we obtain

$$\sum_{\text{cyclic}} \frac{r_b r_c}{m_b m_c} = \sum_{\text{cyclic}} \frac{4s(s-a)}{\sqrt{(2c^2+2a^2-b^2)(2a^2+2b^2-c^2)}}.$$

Applying the AM-GM inequality, we obtain

$$\sum_{\text{cyclic}} \frac{r_b r_c}{m_b m_c} \geq \sum_{\text{cyclic}} \frac{8s(s-a)}{(2c^2+2a^2-b^2)+(2a^2+2b^2-c^2)} = \sum_{\text{cyclic}} \frac{2(a+b+c)(b+c-a)}{4a^2+b^2+c^2}.$$

Thus, it will be enough to show that

$$\sum_{\text{cyclic}} \frac{2(a+b+c)(b+c-a)}{4a^2+b^2+c^2} \geq 3.$$

After expanding the above inequality, we see that it becomes

$$2\sum_{\text{cyclic}} a^6 + 4\sum_{\text{cyclic}} a^4bc + 20\sum_{\text{sym}} a^3b^2c + 68\sum_{\text{cyclic}} a^3b^3 + 16\sum_{\text{cyclic}} a^5b \geq 276a^2b^2c^2 + 27\sum_{\text{cyclic}} a^4b^2.$$

We note that this cannot be proven by just applying Muirhead's Theorem. Since a, b, c are the sides of a triangle, we can make The Ravi Substitution $a = y + z$, $b = z + x$, $c = x + y$, where $x, y, z > 0$. After some brute-force algebra, we can rewrite the above inequality as

$$25\sum_{\text{sym}} x^6 + 230\sum_{\text{sym}} x^5y + 115\sum_{\text{sym}} x^4y^2 + 10\sum_{\text{sym}} x^3y^3 + 80\sum_{\text{sym}} x^4yz$$
$$\geq 336\sum_{\text{sym}} x^3y^2z + 124\sum_{\text{sym}} x^2y^2z^2.$$

Now, by Muirhead's Theorem, we get the result ! $\square$

Epsilon 88. *Let $\mathcal{P}(u,v,w) \in \mathbb{R}[u,v,w]$ be a homogeneous symmetric polynomial with degree 3. Then the following two statements are equivalent.*

(a) $\mathcal{P}(1,1,1),\ \mathcal{P}(1,1,0),\ \mathcal{P}(1,0,0) \geq 0$.
(b) $\mathcal{P}(x,y,z) \geq 0$ for all $x,y,z \geq 0$.

Proof. [SR1] We only prove that (a) implies (b). Let

$$P(u,v,w) = A\sum_{\text{cyclic}} u^3 + B\sum_{\text{sym}} u^2v + Cuvw.$$

Letting $p = P(1,1,1) = 3A+6B+C$, $q = P(1,1,0) = A+B$, and $r = P(1,0,0) = A$, we have $A = r$, $B = q-r$, $C = p-6q+3r$, and $p,q,r \geq 0$. For $x,y,z \geq 0$, we have

$$\begin{aligned} P(x,y,z) &= r\sum_{\text{cyclic}} x^3 + (q-r)\sum_{\text{sym}} x^2y + (p-6q+3r)xyz \\ &= r\left(\sum_{\text{cyclic}} x^3 + 3xyz - \sum_{\text{sym}} x^2y\right) + q\left(\sum_{\text{sym}} x^2y - \sum_{\text{sym}} xyz\right) + pxyz \\ &\geq 0. \end{aligned}$$

□

Remark 8.4. *Here is an alternative way to prove the inequality $P(x,y,z) \geq 0$.*

Case 1. $q \geq r$ *: We compute*

$$P(x,y,z) = \frac{r}{2}\left(\sum_{\text{sym}} x^3 - \sum_{\text{sym}} xyz\right) + (q-r)\left(\sum_{\text{sym}} x^2y - \sum_{\text{sym}} xyz\right) + pxyz.$$

Every term on the right hand side is nonnegative.

Case 2. $q \leq r$ *: We compute*

$$P(x,y,z) = \frac{q}{2}\left(\sum_{\text{sym}} x^3 - \sum_{\text{sym}} xyz\right) + (r-q)\left(\sum_{\text{cyclic}} x^3 + 3xyz - \sum_{\text{sym}} x^2y\right) + pxyz.$$

Every term on the right hand side is nonnegative.

Epsilon 89. [IMO 2001/2 KOR] *Let a, b, c be positive real numbers. Prove that*

$$\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ca}}+\frac{c}{\sqrt{c^2+8ab}}\geq 1.$$

Third Solution. We offer a convexity proof. We make the substitution

$$x=\frac{a}{a+b+c},\ y=\frac{b}{a+b+c},\ z=\frac{c}{a+b+c}.$$

The inequality becomes

$$xf(x^2+8yz)+yf(y^2+8zx)+zf(z^2+8xy)\geq 1,$$

where $f(t)=\frac{1}{\sqrt{t}}$.[22] Since f is convex on $\mathbb{R}^+$ and $x+y+z=1$, we apply (the weighted) Jensen's Inequality to obtain

$$xf(x^2+8yz)+yf(y^2+8zx)+zf(z^2+8xy)\geq f(x(x^2+8yz)+y(y^2+8zx)+z(z^2+8xy)).$$

Note that $f(1)=1$. Since the function f is strictly decreasing, it suffices to show that

$$1\geq x(x^2+8yz)+y(y^2+8zx)+z(z^2+8xy).$$

Using $x+y+z=1$, we homogenize it as

$$(x+y+z)^3\geq x(x^2+8yz)+y(y^2+8zx)+z(z^2+8xy).$$

However, it is easily seen from

$$(x+y+z)^3-x(x^2+8yz)-y(y^2+8zx)-z(z^2+8xy)=3[x(y-z)^2+y(z-x)^2+z(x-y)^2]\geq 0.$$

□

Fourth Solution. We begin with the substitution

$$x=\frac{bc}{a^2}, y=\frac{ca}{b^2}, z=\frac{ab}{c^2}.$$

Then, we get $xyz=1$ and the inequality becomes

$$\frac{1}{\sqrt{1+8x}}+\frac{1}{\sqrt{1+8y}}+\frac{1}{\sqrt{1+8z}}\geq 1$$

which is equivalent to

$$\sum_{\text{cyclic}}\sqrt{(1+8x)(1+8y)}\geq\sqrt{(1+8x)(1+8y)(1+8z)}.$$

After squaring both sides, it's equivalent to

$$8(x+y+z)+2\sqrt{(1+8x)(1+8y)(1+8z)}\sum_{\text{cyclic}}\sqrt{1+8x}\geq 510.$$

Recall that $xyz=1$. The AM-GM Inequality gives us $x+y+z\geq 3$,

$$(1+8x)(1+8y)(1+8z)\geq 9x^{\frac{8}{9}}\cdot 9y^{\frac{8}{9}}\cdot 9z^{\frac{8}{9}}=729 \text{ and } \sum_{\text{cyclic}}\sqrt{1+8x}\geq\sum_{\text{cyclic}}\sqrt{9x^{\frac{8}{9}}}\geq 9(xyz)^{\frac{4}{27}}=9.$$

Using these three inequalities, we get the result. □

[22] Dividing by $a+b+c$ gives the equivalent inequality $\sum_{\text{cyclic}}\frac{\frac{a}{a+b+c}}{\sqrt{\frac{a^2}{(a+b+c)^2}+\frac{8bc}{(a+b+c)^2}}}\geq 1$.

Epsilon 90. [IMO 1983/6 USA] *Let a, b, c be the lengths of the sides of a triangle. Prove that*

$$a^2b(a-b)+b^2c(b-c)+c^2a(c-a)\geq 0.$$

Second Solution. We present a convexity proof. After setting $a=y+z$, $b=z+x$, $c=x+y$ for $x,y,z>0$, it becomes

$$x^3z+y^3x+z^3y\geq x^2yz+xy^2z+xyz^2$$

or

$$\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x}\geq x+y+z.$$

Since it's homogeneous, we can restrict our attention to the case $x+y+z=1$. Then, it becomes

$$yf\left(\frac{x}{y}\right)+zf\left(\frac{y}{z}\right)+xf\left(\frac{z}{x}\right)\geq 1,$$

where $f(t)=t^2$. Since f is convex on $\mathbb{R}$, we apply (the weighted) Jensen's Inequality to obtain

$$yf\left(\frac{x}{y}\right)+zf\left(\frac{y}{z}\right)+xf\left(\frac{z}{x}\right)\geq f\left(y\cdot\frac{x}{y}+z\cdot\frac{y}{z}+x\cdot\frac{z}{x}\right)=f(1)=1.$$

$\square$

Epsilon 91. (KMO Winter Program Test 2001) Prove that, for all $a, b, c > 0$,

$$\sqrt{(a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2)} \geq abc + \sqrt[3]{(a^3 + abc)(b^3 + abc)(c^3 + abc)}$$

First Solution. Dividing by abc, it becomes

$$\sqrt{\left(\frac{a}{c} + \frac{b}{a} + \frac{c}{b}\right)\left(\frac{c}{a} + \frac{a}{b} + \frac{b}{c}\right)} \geq 1 + \sqrt[3]{\left(\frac{a^2}{bc} + 1\right)\left(\frac{b^2}{ca} + 1\right)\left(\frac{c^2}{ab} + 1\right)}.$$

After the substitution $x = \frac{a}{b}$, $y = \frac{b}{c}$, $z = \frac{c}{a}$, we obtain the constraint $xyz = 1$. It takes the form

$$\sqrt{(x + y + z)(xy + yz + zx)} \geq 1 + \sqrt[3]{\left(\frac{x}{z} + 1\right)\left(\frac{y}{x} + 1\right)\left(\frac{z}{y} + 1\right)}.$$

From the constraint $xyz = 1$, we obtain the identity

$$\left(\frac{x}{z} + 1\right)\left(\frac{y}{x} + 1\right)\left(\frac{z}{y} + 1\right) = \left(\frac{x + z}{z}\right)\left(\frac{y + x}{x}\right)\left(\frac{z + y}{y}\right) = (z + x)(x + y)(y + z).$$

Hence, we are required to prove that

$$\sqrt{(x + y + z)(xy + yz + zx)} \geq 1 + \sqrt[3]{(x + y)(y + z)(z + x)}.$$

Now, we offer two ways to finish the proof.

First Method. Observe that

$$(x + y + z)(xy + yz + zx) = (x + y)(y + z)(z + x) + xyz = (x + y)(y + z)(z + x) + 1.$$

Letting $p = \sqrt[3]{(x + y)(y + z)(z + x)}$, the inequality we want to prove now becomes

$$\sqrt{p^3 + 1} \geq 1 + p.$$

Applying The AM-GM Inequality yields

$$p \geq \sqrt[3]{2\sqrt{xy} \cdot 2\sqrt{yz} \cdot 2\sqrt{zx}} = 2.$$

It follows that

$$(p^3 + 1) - (1 + p)^2 = p(p + 1)(p - 2) \geq 0,$$

as desired.

Second Method. More strongly, we establish that, for all $x, y, z > 0$,

$$\sqrt{(x + y + z)(xy + yz + zx)} \geq 1 + \frac{1}{3}\left(\frac{y + z}{\sqrt{yz}} + \frac{z + x}{\sqrt{zx}} + \frac{x + y}{\sqrt{xy}}\right).$$

However, an application of The Cauchy-Schwarz Inequality yields

$$[x + (y + z)][yz + x(y + z)] \geq \left(\sqrt{xyz} + \sqrt{x(y + z)^2}\right)^2 = \left(1 + \frac{y + z}{\sqrt{yz}}\right)^2$$

or

$$\sqrt{(x + y + z)(xy + yz + zx)} \geq 1 + \frac{y + z}{\sqrt{yz}}.$$

Similarly, we also have

$$\sqrt{(x + y + z)(xy + yz + zx)} \geq 1 + \frac{z + x}{\sqrt{zx}}.$$

and

$$\sqrt{(x + y + z)(xy + yz + zx)} \geq 1 + \frac{x + y}{\sqrt{xy}}.$$

Adding these three, we get the desired inequality. $\square$

Epsilon 92. [IMO 1999/2 POL] *Let n be an integer with $n \geq 2$.*

(a) Determine the least constant C such that the inequality

$$\sum_{1 \leq i < j \leq n} x_i x_j \left(x_i^2 + x_j^2\right) \leq C \left(\sum_{1 \leq i \leq n} x_i\right)^4$$

holds for all real numbers $x_1, \cdots, x_n \geq 0$.

(b) For this constant C, determine when equality holds.

First Solution. (Marcin E. Kuczma[23]) For $x_1 = \cdots = x_n = 0$, it holds for any $C \geq 0$. Hence, we consider the case when $x_1 + \cdots + x_n > 0$. Since the inequality is homogeneous, we may normalize to $x_1 + \cdots + x_n = 1$. From the assumption $x_1 + \cdots + x_n = 1$, we have

$$\begin{aligned}
\mathcal{F}(x_1, \cdots, x_n) &= \sum_{1 \leq i < j \leq n} x_i x_j \left(x_i^2 + x_j^2\right) \\
&= \sum_{1 \leq i < j \leq n} {x_i}^3 x_j + \sum_{1 \leq i < j \leq n} x_i {x_j}^3 \\
&= \sum_{1 \leq i \leq n} {x_i}^3 \sum_{j \neq i} x_i \\
&= \sum_{1 \leq i \leq n} {x_i}^3 (1 - x_i) \\
&= \sum_{i=1}^{n} x_i ({x_i}^2 - {x_i}^3).
\end{aligned}$$

We claim that $C = \frac{1}{8}$. It suffices to show that $\mathcal{F}(x_1, \cdots, x_n) \leq \frac{1}{8} = \mathcal{F}\left(\frac{1}{2}, \frac{1}{2}, 0, \cdots, 0\right)$.

Lemma 8.3. $0 \leq x \leq y \leq \frac{1}{2}$ *implies* $x^2 - x^3 \leq y^2 - y^3$.

Proof. Since $x + y \leq 1$, we get $x + y \geq (x + y)^2 \geq x^2 + xy + y^2$. Since $y - x \geq 0$, this implies that $y^2 - x^2 \geq y^3 - x^3$ or $y^2 - y^3 \geq x^2 - x^3$, as desired. □

Case 1. $\frac{1}{2} \geq x_1 \geq x_2 \geq \cdots \geq x_n$*:*

$$\sum_{i=1}^{n} x_i ({x_i}^2 - {x_i}^3) \leq \sum_{i=1}^{n} x_i \left(\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^3\right) = \frac{1}{8} \sum_{i=1}^{n} x_i = \frac{1}{8}.$$

Case 2. $x_1 \geq \frac{1}{2} \geq x_2 \geq \cdots \geq x_n$*: Let $x_1 = x$ and $y = 1 - x = x_2 + \cdots + x_n$. Since $y \geq x_2, \cdots, x_n$,*

$$\mathcal{F}(x_1, \cdots, x_n) = x^3 y + \sum_{i=2}^{n} x_i ({x_i}^2 - {x_i}^3) \leq x^3 y + \sum_{i=2}^{n} x_i (y^2 - y^3) = x^3 y + y(y^2 - y^3).$$

Since $x^3 y + y(y^2 - y^3) = x^3 y + y^3(1 - y) = xy(x^2 + y^2)$, it remains to show that

$$xy(x^2 + y^2) \leq \frac{1}{8}.$$

Using $x + y = 1$, we homogenize the above inequality as following.

$$xy(x^2 + y^2) \leq \frac{1}{8}(x + y)^4.$$

However, we immediately find that $(x + y)^4 - 8xy(x^2 + y^2) = (x - y)^4 \geq 0$.

□

[23] I slightly modified his solution in [AS].

Epsilon 93. (APMO 1991) Let $a_1, \cdots, a_n, b_1, \cdots, b_n$ be positive real numbers such that $a_1 + \cdots + a_n = b_1 + \cdots + b_n$. Show that

$$\frac{{a_1}^2}{a_1 + b_1} + \cdots + \frac{{a_n}^2}{a_n + b_n} \geq \frac{a_1 + \cdots + a_n}{2}.$$

Second Solution. By The Cauchy-Schwarz Inequality, we have

$$\left(\sum_{i=1}^{n} a_i + b_i\right)\left(\sum_{i=1}^{n} \frac{{a_i}^2}{a_i + b_i}\right) \geq \left(\sum_{i=1}^{n} a_i\right)^2$$

or

$$\sum_{i=1}^{n} \frac{{a_i}^2}{a_i + b_i} \geq \frac{\left(\sum_{i=1}^{n}\right)^2}{\sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i} = \frac{1}{2}\sum_{i=1}^{n} a_i$$

$\square$

Epsilon 94. *Let $a, b \geq 0$ with $a + b = 1$. Prove that*

$$\sqrt{a^2+b} + \sqrt{a+b^2} + \sqrt{1+ab} \leq 3.$$

Show that the equality holds if and only if $(a, b) = (1, 0)$ or $(a, b) = (0, 1)$.

Second Solution. The Cauchy-Schwarz Inequality shows that

$$\begin{aligned}
\sqrt{a^2+b} + \sqrt{a+b^2} + \sqrt{1+ab} &\leq \sqrt{3\left(a^2+b+a+b^2+1+ab\right)} \\
&= \sqrt{3\left(a^2+ab+b^2+a+b+1\right)} \\
&\leq \sqrt{3\left((a+b)^2+a+b+1\right)} \\
&= 3.
\end{aligned}$$

$\square$

Epsilon 95. [LL 1992 UNK] (Iran 1998) Prove that, for all $x, y, z > 1$ such that $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=2$,

$$\sqrt{x+y+z} \geq \sqrt{x-1}+\sqrt{y-1}+\sqrt{z-1}.$$

Third Solution. We first note that

$$\frac{x-1}{x}+\frac{y-1}{y}+\frac{z-1}{z}=1.$$

Apply The Cauchy-Schwarz Inequality to deduce

$$\sqrt{x+y+z}=\sqrt{(x+y+z)\left(\frac{x-1}{x}+\frac{y-1}{y}+\frac{z-1}{z}\right)} \geq \sqrt{x-1}+\sqrt{y-1}+\sqrt{z-1}.$$

$\square$

Epsilon 96. (Gazeta Matematicã) Prove that, for all $a, b, c > 0$,

$$\sqrt{a^4+a^2b^2+b^4}+\sqrt{b^4+b^2c^2+c^4}+\sqrt{c^4+c^2a^2+a^4} \geq a\sqrt{2a^2+bc}+b\sqrt{2b^2+ca}+c\sqrt{2c^2+ab}.$$

Solution. We obtain the chain of equalities and inequalities

$$\begin{aligned}
\sum_{\text{cyclic}} \sqrt{a^4+a^2b^2+b^4} &= \sum_{\text{cyclic}} \sqrt{\left(a^4+\frac{a^2b^2}{2}\right)+\left(b^4+\frac{a^2b^2}{2}\right)} \\
&\geq \frac{1}{\sqrt{2}} \sum_{\text{cyclic}} \left(\sqrt{a^4+\frac{a^2b^2}{2}}+\sqrt{b^4+\frac{a^2b^2}{2}}\right) \qquad (\text{Cauchy}-\text{Schwarz}) \\
&= \frac{1}{\sqrt{2}} \sum_{\text{cyclic}} \left(\sqrt{a^4+\frac{a^2b^2}{2}}+\sqrt{a^4+\frac{a^2c^2}{2}}\right) \\
&\geq \sqrt{2} \sum_{\text{cyclic}} \sqrt[4]{\left(a^4+\frac{a^2b^2}{2}\right)\left(a^4+\frac{a^2c^2}{2}\right)} \qquad (\text{AM}-\text{GM}) \\
&\geq \sqrt{2} \sum_{\text{cyclic}} \sqrt{a^4+\frac{a^2bc}{2}} \qquad (\text{Cauchy}-\text{Schwarz}) \\
&= \sum_{\text{cyclic}} \sqrt{2a^4+a^2bc}\,.
\end{aligned}$$

□

Epsilon 97. (KMO Winter Program Test 2001) Prove that, for all $a, b, c > 0$,

$$\sqrt{\left(a^2b+b^2c+c^2a\right)\left(ab^2+bc^2+ca^2\right)} \geq abc + \sqrt[3]{\left(a^3+abc\right)\left(b^3+abc\right)\left(c^3+abc\right)}$$

Second Solution. (based on work by an winter program participant) We obtain

$$\begin{aligned}
& \sqrt{\left(a^2b+b^2c+c^2a\right)\left(ab^2+bc^2+ca^2\right)} \\
= \; & \frac{1}{2}\sqrt{\left[b(a^2+bc)+c(b^2+ca)+a(c^2+ab)\right]\left[c(a^2+bc)+a(b^2+ca)+b(c^2+ab)\right]} \\
\geq \; & \frac{1}{2}\left(\sqrt{bc}(a^2+bc)+\sqrt{ca}(b^2+ca)+\sqrt{ab}(c^2+ab)\right) && \text{(Cauchy − Schwarz)} \\
\geq \; & \frac{3}{2}\sqrt[3]{\sqrt{bc}(a^2+bc)\cdot\sqrt{ca}(b^2+ca)\cdot\sqrt{ab}(c^2+ab)} && \text{(AM − GM)} \\
= \; & \frac{1}{2}\sqrt[3]{\left(a^3+abc\right)\left(b^3+abc\right)\left(c^3+abc\right)} + \sqrt[3]{\left(a^3+abc\right)\left(b^3+abc\right)\left(c^3+abc\right)} \\
\geq \; & \frac{1}{2}\sqrt[3]{2\sqrt{a^3\cdot abc}\cdot 2\sqrt{b^3\cdot abc}\cdot 2\sqrt{c^3\cdot abc}} + \sqrt[3]{\left(a^3+abc\right)\left(b^3+abc\right)\left(c^3+abc\right)} && \text{(AM − GM)} \\
= \; & abc + \sqrt[3]{\left(a^3+abc\right)\left(b^3+abc\right)\left(c^3+abc\right)}.
\end{aligned}$$

□

Epsilon 98. (Andrei Ciupan, Romanian Junior Balkan MO 2007 Team Selection Tests) Let a, b, c be positive real numbers such that

$$\frac{1}{a+b+1}+\frac{1}{b+c+1}+\frac{1}{c+a+1}\geq 1.$$

Show that $a+b+c\geq ab+bc+ca$.

First Solution. By applying The Cauchy-Schwarz Inequality, we obtain

$$(a+b+1)(a+b+c^2)\geq (a+b+c)^2$$

or

$$\frac{1}{a+b+1}\leq \frac{c^2+a+b}{(a+b+c)^2}.$$

Now by summing cyclically, we obtain

$$\frac{1}{a+b+1}+\frac{1}{b+c+1}+\frac{1}{c+a+1}\leq \frac{a^2+b^2+c^2+2(a+b+c)}{(a+b+c)^2}$$

But from the condition, we can see that

$$a^2+b^2+c^2+2(a+b+c)\geq (a+b+c)^2,$$

and therefore

$$a+b+c\geq ab+bc+ca.$$

We see that the equality occurs if and only if $a=b=c=1$. $\square$

Second Solution. (Cezar Lupu) We first observe that

$$2\geq \sum_{\text{cyclic}}\left(1-\frac{1}{a+b+1}\right)=\sum_{\text{cyclic}}\frac{a+b}{a+b+1}=\sum_{\text{cyclic}}\frac{(a+b)^2}{(a+b)^2+a+b}.$$

Apply The Cauchy-Schwarz Inequality to get

$$\begin{aligned}
2 &\geq \sum_{\text{cyclic}}\frac{(a+b)^2}{(a+b)^2+a+b}\\
&\geq \frac{\left(\sum_{\text{cyclic}}a+b\right)^2}{\sum_{\text{cyclic}}(a+b)^2+a+b}\\
&= \frac{4\sum_{\text{cyclic}}a^2+8\sum_{\text{cyclic}}ab}{2\sum_{\text{cyclic}}a^2+2\sum_{\text{cyclic}}ab+2\sum_{\text{cyclic}}a}
\end{aligned}$$

or

$$a+b+c\geq ab+bc+ca.$$

$\square$

Epsilon 99. (Hölder's Inequality) Let x_{ij} $(i = 1, \cdots, m, j = 1, \cdots n)$ be positive real numbers. Suppose that $\omega_1, \cdots, \omega_n$ are positive real numbers satisfying $\omega_1 + \cdots + \omega_n = 1$. Then, we have

$$\prod_{j=1}^{n} \left(\sum_{i=1}^{m} x_{ij} \right)^{\omega_j} \geq \sum_{i=1}^{m} \left(\prod_{j=1}^{n} {x_{ij}}^{\omega_j} \right).$$

Proof. Because of the homogeneity of the inequality, we may rescale $x_{1j}, \cdots, x_{mj}$ so that $x_{1j} + \cdots + x_{mj} = 1$ for each $j \in \{1, \cdots, n\}$. Then, we need to show that

$$\prod_{j=1}^{n} 1^{\omega_j} \geq \sum_{i=1}^{m} \prod_{j=1}^{n} {x_{ij}}^{\omega_j} \quad \text{or} \quad 1 \geq \sum_{i=1}^{m} \prod_{j=1}^{n} {x_{ij}}^{\omega_j}.$$

The Weighted AM-GM Inequality provides that

$$\sum_{j=1}^{n} \omega_j x_{ij} \geq \prod_{j=1}^{n} {x_{ij}}^{\omega_j} \quad (i \in \{1, \cdots, m\}) \implies \sum_{i=1}^{m} \sum_{j=1}^{n} \omega_j x_{ij} \geq \sum_{i=1}^{m} \prod_{j=1}^{n} {x_{ij}}^{\omega_j}.$$

However, we immediately have

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \omega_j x_{ij} = \sum_{j=1}^{n} \sum_{i=1}^{m} \omega_j x_{ij} = \sum_{j=1}^{n} \omega_j \left(\sum_{i=1}^{m} x_{ij} \right) = \sum_{j=1}^{n} \omega_j = 1.$$

$\square$

Epsilon 100. *Let $f : [a,b] \longrightarrow \mathbb{R}$ be a* **continuous** *function. Then, the followings are equivalent.*

(1) For all $n \in \mathbb{N}$, the following inequality holds.

$$\omega_1 f(x_1) + \cdots + \omega_n f(x_n) \geq f(\omega_1\, x_1 + \cdots + \omega_n\, x_n)$$

for all $x_1, \cdots, x_n \in [a,b]$ and $\omega_1, \cdots, \omega_n > 0$ with $\omega_1 + \cdots + \omega_n = 1$.
(2) For all $n \in \mathbb{N}$, the following inequality holds.

$$r_1 f(x_1) + \cdots + r_n f(x_n) \geq f(r_1\, x_1 + \cdots + r_n\, x_n)$$

for all $x_1, \cdots, x_n \in [a,b]$ and $r_1, \cdots, r_n \in \mathbb{Q}^+$ with $r_1 + \cdots + r_n = 1$.
(3) For all $N \in \mathbb{N}$, the following inequality holds.

$$\frac{f(y_1) + \cdots + f(y_N)}{N} \geq f\left(\frac{y_1 + \cdots + \ y_N}{N}\right)$$

for all $y_1, \cdots, y_N \in [a,b]$.
(4) For all $k \in \{0,1,2,\cdots\}$, the following inequality holds.

$$\frac{f(y_1) + \cdots + f(y_{2^k})}{2^k} \geq f\left(\frac{y_1 + \cdots + \ y_{2^k}}{2^k}\right)$$

for all $y_1, \cdots, y_{2^k} \in [a,b]$.
(5) We have $\frac{1}{2}f(x) + \frac{1}{2}f(y) \geq f\left(\frac{x+y}{2}\right)$ for all $x, y \in [a,b]$.
(6) We have $\lambda f(x) + (1-\lambda)f(y) \geq f\left(\lambda x + (1-\lambda)y\right)$ for all $x, y \in [a,b]$ and $\lambda \in (0,1)$.

Solution. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) is obvious.

(2) $\Rightarrow$ (1) **:** Let $x_1, \cdots, x_n \in [a,b]$ and $\omega_1, \cdots, \omega_n > 0$ with $\omega_1 + \cdots + \omega_n = 1$. One may see that there exist positive rational sequences $\{r_k(1)\}_{k\in\mathbb{N}}, \cdots, \{r_k(n)\}_{k\in\mathbb{N}}$ satisfying

$$\lim_{k\to\infty} r_k(j) = w_j \ \ (1 \leq j \leq n) \ \text{ and } \ r_k(1) + \cdots + r_k(n) = 1 \ \text{ for all } \ k \in \mathbb{N}.$$

By the hypothesis in (2), we obtain $r_k(1)f(x_1) + \cdots + r_k(n)f(x_n) \geq f(r_k(1)\, x_1 + \cdots + r_k(n)\, x_n)$. Since f is continuous, taking $k \to \infty$ to both sides yields the inequality

$$\omega_1 f(x_1) + \cdots + \omega_n f(x_n) \geq f(\omega_1\, x_1 + \cdots + \omega_n\, x_n).$$

(3) $\Rightarrow$ (2) **:** Let $x_1, \cdots, x_n \in [a,b]$ and $r_1, \cdots, r_n \in \mathbb{Q}^+$ with $r_1 + \cdots + r_n = 1$. We can find a positive integer $N \in \mathbb{N}$ so that $Nr_1, \cdots, Nr_n \in \mathbb{N}$. For each $i \in \{1, \cdots, n\}$, we can write $r_i = \frac{p_i}{N}$, where $p_i \in \mathbb{N}$. It follows from $r_1 + \cdots + r_n = 1$ that $N = p_1 + \cdots + p_n$. Then, (3) implies that

$$\begin{aligned}
& r_1 f(x_1) + \cdots + r_n f(x_n) \\
= \ & \frac{\overbrace{f(x_1) + \cdots + f(x_1)}^{p_1 \text{ terms}} + \cdots + \overbrace{f(x_n) + \cdots + f(x_n)}^{p_n \text{ terms}}}{N} \\
\geq \ & f\left(\frac{\overbrace{x_1 + \cdots + x_1}^{p_1 \text{ terms}} + \cdots + \overbrace{x_n + \cdots + x_n}^{p_n \text{ terms}}}{N}\right) \\
= \ & f(r_1\, x_1 + \cdots + r_n\, x_n).
\end{aligned}$$

(4) $\Rightarrow$ (3) **:** Let $y_1, \cdots, y_N \in [a,b]$. Take a large $k \in \mathbb{N}$ so that $2^k > N$. Let $a = \frac{y_1+\cdots+y_N}{N}$. Then, (4) implies that

$$\begin{aligned}
& \frac{f(y_1)+\cdots+f(y_N)+(2^k-n)f(a)}{2^k} \\
= \quad & \frac{f(y_1)+\cdots+f(y_N)+\overbrace{f(a)+\cdots+f(a)}^{(2^k-N)\text{ terms}}}{2^k} \\
\geq \quad & f\left(\frac{y_1+\cdots+y_N+\overbrace{a+\cdots+a}^{(2^k-N)\text{ terms}}}{2^k}\right) \\
= \quad & f(a)
\end{aligned}$$

so that

$$f(y_1)+\cdots+f(y_N) \geq Nf(a) = Nf\left(\frac{y_1+\cdots+y_N}{N}\right).$$

(5) $\Rightarrow$ (4) **:** We use induction on k. In case $k = 0, 1, 2$, it clearly holds. Suppose that (4) holds for some $k \geq 2$. Let $y_1, \cdots, y_{2^{k+1}} \in [a,b]$. By the induction hypothesis, we obtain

$$\begin{aligned}
& f(y_1)+\cdots+f(y_{2^k})+f(y_{2^k+1})+\cdots+f(y_{2^{k+1}}) \\
\geq \quad & 2^k f\left(\frac{y_1+\cdots+\ y_{2^k}}{2^k}\right)+2^k f\left(\frac{y_{2^k+1}+\cdots+\ y_{2^{k+1}}}{2^k}\right) \\
= \quad & 2^{k+1}\frac{f\left(\frac{y_1+\cdots+\ y_{2^k}}{2^k}\right)+f\left(\frac{y_{2^k+1}+\cdots+\ y_{2^{k+1}}}{2^k}\right)}{2} \\
\geq \quad & 2^{k+1} f\left(\frac{\frac{y_1+\cdots+\ y_{2^k}}{2^k}+\frac{y_{2^k+1}+\cdots+\ y_{2^{k+1}}}{2^k}}{2}\right) \\
= \quad & 2^{k+1} f\left(\frac{y_1+\cdots+\ y_{2^{k+1}}}{2^{k+1}}\right).
\end{aligned}$$

Hence, (4) holds for $k+1$. This completes the induction.

So far, we've established that (1), (2), (3), (4), (5) are all equivalent. Since (1) $\Rightarrow$ (6) $\Rightarrow$ (5) is obvious, this completes the proof. $\square$

Epsilon 101. *Let x, y, z be nonnegative real numbers. Then, we have*

$$3xyz + x^3 + y^3 + z^3 \geq 2\left((xy)^{\frac{3}{2}} + (yz)^{\frac{3}{2}} + (zx)^{\frac{3}{2}}\right).$$

Second Solution. After employing the substitution

$$x = e^{\frac{p}{3}},\ y = e^{\frac{q}{3}},\ z = e^{\frac{r}{3}},$$

the inequality becomes

$$3e^{\frac{p+q+r}{3}} + e^p + e^q + e^r \geq 2\left(e^{\frac{q+r}{2}} + e^{\frac{r+p}{2}} + e^{\frac{p+q}{2}}\right)$$

It is a straightforward consequence of Popoviciu's Inequality. $\square$

Epsilon 102. *Let ABC be an acute triangle. Show that*

$$\cos A + \cos B + \cos C \geq 1.$$

Proof. Observe that $\left(\frac{\pi}{2}, \frac{\pi}{2}, 0\right)$ majorize (A, B, C). Since $-cosx$ is convex on $\left(0, \frac{\pi}{2}\right)$, The Hardy-Littlewood-Pólya Inequality implies that

$$\cos A + \cos B + \cos C \geq \cos\left(\frac{\pi}{2}\right) + \cos\left(\frac{\pi}{2}\right) + \cos 0 = 1.$$

$\square$

Epsilon 103. *Let ABC be a triangle. Show that*

$$\tan^2\left(\frac{A}{4}\right)+\tan^2\left(\frac{B}{4}\right)+\tan^2\left(\frac{C}{4}\right)\leq 1.$$

Proof. Observe that $(\pi,0,0)$ majorizes (A,B,C). The convexity of $\tan^2\left(\frac{x}{4}\right)$ on $[0,\pi]$ yields the estimation:

$$\tan^2\left(\frac{A}{4}\right)+\tan^2\left(\frac{B}{4}\right)+\tan^2\left(\frac{C}{4}\right)\leq \tan^2\left(\frac{\pi}{4}\right)+\tan^2 0+\tan^2 0=1.$$

□

Epsilon 104. *Use The Hardy-Littlewood-Pólya Inequality to deduce Popoviciu's Inequality.*

Proof. [NP, p.33] Since the inequality is symmetric, we may assume that $x \geq y \geq z$. We consider the two cases. In the case when $x \geq \frac{x+y+z}{3} \geq y \geq z$, the majorization

$$\left(x, \frac{x+y+z}{3}, \frac{x+y+z}{3}, \frac{x+y+z}{3}, y, z\right) \succ \left(\frac{x+y}{2}, \frac{x+y}{2}, \frac{z+x}{2}, \frac{z+x}{2}, \frac{y+z}{2}, \frac{y+z}{2}\right)$$

yields Popoviciu's Inequality. In the case when $x \geq y \geq \frac{x+y+z}{3} \geq z$, the majorization

$$\left(x, y, \frac{x+y+z}{3}, \frac{x+y+z}{3}, \frac{x+y+z}{3}, z\right) \succ \left(\frac{x+y}{2}, \frac{x+y}{2}, \frac{z+x}{2}, \frac{z+x}{2}, \frac{y+z}{2}, \frac{y+z}{2}\right)$$

yields Popoviciu's Inequality. $\square$

Epsilon 105. [IMO 1999/2 POL] *Let n be an integer with $n \geq 2$.*

Determine the least constant C such that the inequality

$$\sum_{1\leq i<j\leq n} x_i x_j \left(x_i^2 + x_j^2\right) \leq C \left(\sum_{1\leq i\leq n} x_i\right)^4$$

holds for all real numbers $x_1, \cdots, x_n \geq 0$.

Second Solution. (Kin Y. Li[24]) According to the homogenity of the inequality, we may normalize to $x_1 + \cdots + x_n = 1$. Our job is to maximize

$$\begin{aligned}
\mathcal{F}(x_1, \cdots, x_n) &= \sum_{1\leq i<j\leq n} x_i x_j \left(x_i^2 + x_j^2\right) \\
&= \sum_{1\leq i<j\leq n} {x_i}^3 x_j + \sum_{1\leq i<j\leq n} x_i {x_j}^3 \\
&= \sum_{1\leq i\leq n} {x_i}^3 \sum_{j\neq i} x_i \\
&= \sum_{1\leq i\leq n} {x_i}^3 (1 - x_i) \\
&= \sum_{i=1}^{n} f(x_i),
\end{aligned}$$

where $f(t) = t^3 - t^4$ is a convex function on $\left[0, \frac{1}{2}\right]$. Since the inequality is symmetric, we can restrict our attention to the case $x_1 \geq x_2 \geq \cdots \geq x_n$. If $\frac{1}{2} \geq x_1$, then we see that $\left(\frac{1}{2}, \frac{1}{2}, 0, \cdots 0\right)$ majorizes $(x_1, \cdots, x_n)$. Since $x_1, \cdots, x_2, \cdots, x_n \in \left[0, \frac{1}{2}\right]$ and since f is convex on $\left[0, \frac{1}{2}\right]$, by The Hardy-Littlewood-Pólya Inequality, the convexity of f on $[0, \frac{1}{2}]$ implies that

$$\sum_{i=1}^{n} f(x_i) \leq f\left(\frac{1}{2}\right) + f\left(\frac{1}{2}\right) + f(0) + \cdots + f(0) = \frac{1}{8}.$$

We now consider the case when $x_1 \geq \frac{1}{2}$. We find that $(1 - x_1, 0, \cdots 0)$ majorizes $(x_2, \cdots, x_n)$. Since $1 - x_1, x_2, \cdots, x_n \in \left[0, \frac{1}{2}\right]$ and since f is convex on $\left[0, \frac{1}{2}\right]$, by The Hardy-Littlewood-Pólya Inequality,

$$\sum_{i=2}^{n} f(x_i) \leq f(1 - x_1) + f(0) + \cdots + f(0) = f(1 - x_1).$$

Setting $x_1 = \frac{1}{2} + \epsilon$ for some $\epsilon \in \left[0, \frac{1}{2}\right]$, we obtain

$$\begin{aligned}
\sum_{i=1}^{n} f(x_i) &\leq f(x_1) + f(1 - x_1) \\
&= x_1(1 - x_1)[{x_1}^2 + (1 - x_1)^2] \\
&= \left(\frac{1}{4} - \epsilon^2\right)\left(\frac{1}{2} + 2\epsilon^2\right) \\
&= 2\left(\frac{1}{16} - \epsilon^4\right) \\
&\leq \frac{1}{8}.
\end{aligned}$$

□

[24] I slightly modified his solution in [KL].

9. Appendix

9.1. References.

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9.2. **IMO Code.**

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AFG	Afghanistan	ALB	Albania	ALG	Algeria
ARG	Argentina	ARM	Armenia	AUS	Australia
AUT	Austria	AZE	Azerbaijan	BAH	Bahrain
BGD	Bangladesh	BLR	Belarus	BEL	Belgium
BEN	Benin	BOL	Bolivia	BIH	BIH
BRA	Brazil	BRU	Brunei	BGR	Bulgaria
KHM	Cambodia	CMR	Cameroon	CAN	Canada
CHI	Chile	CHN	CHN	COL	Colombia
CIS	CIS	CRI	Costa Rica	HRV	Croatia
CUB	Cuba	CYP	Cyprus	CZE	Czech Republic
CZS	Czechoslovakia	DEN	Denmark	DOM	Dominican Republic
ECU	Ecuador	EST	Estonia	FIN	Finland
FRA	France	GEO	Georgia	GDR	GDR
GER	Germany	HEL	Greece	GTM	Guatemala
HND	Honduras	HKG	Hong Kong	HUN	Hungary
ISL	Iceland	IND	India	IDN	Indonesia
IRN	Islamic Republic of Iran	IRL	Ireland	ISR	Israel
ITA	Italy	JPN	Japan	KAZ	Kazakhstan
PRK	PRK	KOR	Republic of Korea	KWT	Kuwait
KGZ	Kyrgyzstan	LVA	Latvia	LIE	Liechtenstein
LTU	Lithuania	LUX	Luxembourg	MAC	Macau
MKD	MKD	MAS	Malaysia	MLT	Malta
MRT	Mauritania	MEX	Mexico	MDA	Republic of Moldova
MNG	Mongolia	MNE	Montenegro	MAR	Morocco
MOZ	Mozambique	NLD	Netherlands	NZL	New Zealand
NIC	Nicaragua	NGA	Nigeria	NOR	Norway
PAK	Pakistan	PAN	Panama	PAR	Paraguay
PER	Peru	PHI	Philippines	POL	Poland
POR	Portugal	PRI	Puerto Rico	ROU	Romania
RUS	Russian Federation	SLV	El Salvador	SAU	Saudi Arabia
SEN	Senegal	SRB	Serbia	SCG	Serbia and Montenegro
SGP	Singapore	SVK	Slovakia	SVN	Slovenia
SAF	South Africa	ESP	Spain	LKA	Sri Lanka
SWE	Sweden	SUI	Switzerland	SYR	Syria
TWN	Taiwan	TJK	Tajikistan	THA	Thailand
TTO	Trinidad and Tobago	TUN	Tunisia	TUR	Turkey
NCY	NCY	TKM	Turkmenistan	UKR	Ukraine
UAE	United Arab Emirates	UNK	United Kingdom	USA	United States of America
URY	Uruguay	USS	USS	UZB	Uzbekistan
VEN	Venezuela	VNM	Vietnam	YUG	Yugoslavia

BIH	Bosnia and Herzegovina
CHN	People's Republic of China
CIS	Commonwealth of Independent States
FRG	Federal Republic of Germany
GDR	German Democratic Republic
MKD	The Former Yugoslav Republic of Macedonia
NCY	Turkish Republic of Northern Cyprus
PRK	Democratic People's Republic of Korea
USS	Union of the Soviet Socialist Republics