# CALCULUS II

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## Lecture 3

## Infinite Series

In this lecture we are going to study sums of infinite sequences $(a_n)$ of real numbers (usually $(a_n)_{n\geq 1}$), i.e.

$$a_1 + a_2 + \ldots + a_n + \ldots .$$

The question arises how to define such an infinite sum in a meaningful way and develop some theory which would tell us when such a sum is finite or infinite. For that purpose, we start with the following definitions.

Definition 1. Let $(a_n)_{n\geq 1}$ be a sequence of real numbers. The sum

$$S_n = a_1 + \ldots + a_n$$

is called the *$n$-th partial sum*. By an *infinite series* we understand the sequence $(S_n)_{n\geq 1}$. If $S = \lim_{n\to\infty} S_n$ exists and is finite we say that the series *converges* to $S$ and if $S = \infty$ or $S = -\infty$, then we say that it *diverges to* $\infty$ or $-\infty$, respectively. The limit $S$ is then called the *sum* of the series. If the limit of partial sums doesn't exist, we say that the series *diverges*. It is convenient to denote infinite series as

$$\sum_{n=1}^{\infty} a_n$$

and let the context show whether the expression represents the series itself or the sum of all its terms. In an analogous fashion we define series of the form $\sum_{n=n_0}^{\infty} a_n$.

Example 1. Using Definition 1, test for convergence or divergence the given series. In the series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

we can find an explicit expression for the $n$-th partial sum if we notice that

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

since then we have

$$S_n = (\frac{1}{1} - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \ldots + (\frac{1}{n} - \frac{1}{n+1}) = 1 - \frac{1}{n+1} \to 1$$

and thus the series converges to 1.

Consider now

$$\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n}\right)$$

Note that $a_n = \ln(1+1/n) = \ln(n+1)/n = \ln(n+1) - \ln n$ which gives $S_n = \ln n \to \infty$ and hence our series diverges to $\infty$.

An easy example of a divergent series is given by

$$\sum_{n=1}^{\infty} (-1)^n$$

since

$$S_n = \begin{cases} -1 & \text{if} \quad n = 2k-1 \\ 0 & \text{if} \quad n = 2k \end{cases}$$

and hence $\lim_{n\to\infty} S_n$ doesn't exist (it has two subsequences converging to two diferent numbers) and the given series diverges.

Proposition 1. *The geometric series*

$$\sum_{n=0}^{\infty} q^n$$

*converges to* $1/(1-q)$ *if* $|q| < 1$, *diverges to* $\infty$ *if* $q \geq 1$ *and otherwise diverges.*

*Proof.* Let us give a proof of this fact. We know that

$$S_n = 1 + q + q^2 + \ldots + q^n = \frac{1 - q^{n+1}}{1 - q}$$

if $q \neq 1$ and therefore, if $|q| < 1$, then $S_n \to 1/(1-q)$. If $q > 1$, then $1 - q^{n+1} \to \infty$ and thus $S_n \to \infty$. If $q = 1$, then we get $S_n = n \to \infty$. Finally, in the remaining cases we see that the limit $\lim_{n\to\infty}(1 - q^{n+1})$ doesn't exist, hence the series diverges. □

Proposition 2. *If the series* $\sum_{n=1}^{\infty} a_n$ *converges then* $\lim_{n\to\infty} a_n = 0$

*Proof.* Suppose that $\lim_{n\to\infty} a_n \neq 0$. Then

$$S_{n+1} - S_n = a_{n+1}$$

and if $S_n$ converged, say to $S$, then we would have

$$\lim_{n\to\infty} (S_{n+1} - S_n) = S - S = 0$$

where which is a contradiction. This proves our implication. $\square$

The above proposition gives a necessary convergence condition which is sometimes useful. For instance, we can easily see that the series

$$\sum_{n=1}^{\infty} n \sin \frac{1}{n}$$

does not converge since $\lim_{x \to 0} \sin x / x = 1$ and the necessary condition of Proposition 2 is not satisfied. In fact, it is also easy to see that this series diverges to $\infty$ since all its terms are positive.

THEOREM 1. (INTEGRAL TEST). *Suppose that a series* $\sum_{n=n_0}^{\infty} a_n$ *is given where* $a_n = f(n)$ *and* $f$ *is a continuous, nonnegative and nonincreasing function on* $[n_0, \infty)$, *where* $n_0 \in \mathbf{N}$. *Then*

(a) *the series* $\sum_{n=n_0}^{\infty} a_n$ *converges if the integral* $\int_{n_0}^{\infty} f(x) dx$ *converges*

(b) *the series* $\sum_{n=n_0}^{\infty} a_n$ *diverges to* $\infty$ *if the integral* $\int_{n_0}^{\infty} f(x) dx$ *diverges to* $\infty$

Note that since all the terms of the series are nonnegative, the cases of a divergent series or a series divergent to $-\infty$ are ruled out.

EXAMPLE 2. Let us consider the series

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

First, let us notice that if we take $f(x) = 1/(x \ln x)$, then the assumptions of the integral test are satisfied. Namely, $f$ strictly decreases over $[2, \infty)$. Moreover, in the integral $\int_2^{\infty} 1/(x \ln x) dx$ we make the substitution $\ln x = t$ which gives

$$\int_2^{\infty} \frac{dx}{x \ln x} = \int_{\ln 2}^{\infty} \frac{dt}{t} = \infty$$

hence, by Theorem 1 (b), our series diverges.

PROPOSITION 3. *The series*

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

*converges for* $p > 1$ *and diverges to* $\infty$ *for* $p \leq 1$.

*Proof.* It is a straightforward consequence of the integral test. Note that if $p \geq 0$, the assumptions of the integral test are satisfied. Now, let $p > 1$. Since the integral $\int_1^{\infty} dx/x^p$ converges for $p < 1$, the series also converges. Similarly, if $0 \leq p \leq 1$, the series diverges to $\infty$ since the corresponding integral diverges to $\infty$. If $p < 0$, then it is clear that the series diverges to $\infty$ since all its terms are greater or equal to 1. $\square$

THEOREM 2. (COMPARISON TEST) *Suppose that* $0 \le a_n \le b_n$ *for all* $n \ge n_0$. *Then*

(a) *if* $\sum_{n=n_0}^{\infty} b_n$ *converges, then* $\sum_{n=n_0}^{\infty}$ *converges*

(b) *if* $\sum_{n=n_0}^{\infty} a_n$ *diverges to* $\infty$, *then* $\sum_{n=1}^{\infty} b_n$ *diverges to* $\infty$.

EXAMPLE 3. Below we use a short-hand notation, but filling in the details and checking the assumptions is easy (the reader is advised to do that).

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+1} > \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+\sqrt{n}} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \infty$$

by Proposition 3 ($p = 1/2 < 1$).

$$\sum_{n=1}^{\infty} n \sin \frac{1}{n^3} < \sum_{n=1}^{\infty} n \frac{1}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

by Proposition 3 again (here, $p = 2 > 1$). We used the inequality $\sin x \le x$ which holds for all $x \ge 0$.

THEOREM 3. (LIMIT RATIO TEST) *Suppose that* $a_n, b_n > 0$ *for all* $n \ge n_0$ *and that*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = K$$

*where* $0 < K < \infty$. *Then*

(a) $\sum_{n=1}^{\infty} a_n$ *converges if and only if* $\sum_{n=1}^{\infty} b_n$ *converges*

(b) $\sum_{n=1}^{\infty} a_n$ *diverges to* $\infty$ *if and only if* $\sum_{n=1}^{\infty} b_n$ *diverges to* $\infty$

*Remark.* A similar theorem holds if $a_n, b_n < 0$ for all $n \ge n_0$. In that case, in (b) we have divergence to $-\infty$ instead of $\infty$. Note also that the assumption that all terms, except perhaps a finite number of them, are positive or all are negative implies that such series cannot diverge in the snse of Definition 1.

EXAMPLE 4. Test for convergence the series

$$\sum_{n=1}^{\infty} \arcsin \frac{1}{n}.$$

Let

$$a_n = \arcsin \frac{1}{n}, \quad b_n = \frac{1}{n}$$

Then $a_n, b_n > 0$ for all $n \ge 1$ and

$$\lim_{n\to\infty} \frac{\arcsin 1/n}{1/n} = 1$$

hence $K = 1$ and all assumptions of Theorem 3 are satisfied. Therefore, the series diverges to $\infty$ since the harmonic series $\sum_{n=1}^{\infty} 1/n$ diverges to $\infty$.

EXAMPLE 5. Test for convergence the series

$$\sum_{n=1}^{\infty} \frac{n+2}{n^3-n^2+n-1}.$$

It is a little cumbersome to find a specific $n_0$ of Theorem 3, but it is clear that such $n_0$ such that

$$a_n = \frac{n+2}{n^3-n^2+n-1} > 0$$

for all $n \geq n_0$ since the numerator is always positive and the denominator is positive for large $n$ since the leading power of $n$ has a positive coefficient. Since the numerator behaves like $n$ for large $n$ and the denominator behaves like $n^3$ for large $n$, we guess that a good choice for $b_n$ is $b_n = 1/n^2$ which gives a convergent series $\sum_{n=1}^{\infty} 1/n^2$, thus, by Theorem 3, our series is also convergent.

THEOREM 4. (CAUCHY'S TEST) *Suppose that* $\lim_{n\to\infty} |a_n|^{1/n} = q$, *where* $q \geq 0$, *by which we mean that the limit exists and is equal* $q$, *allowing also* $q = \infty$. *Then the series* $\sum_{n=1}^{\infty} a_n$ *converges if* $q < 1$ *and it does not converge if* $q > 1$.

*Proof.* We will prove the theorem in the case when $a_n \geq 0$ for large $n$. If $q < 1$, then for each $\epsilon < 1 - q$ there exists $n_0 \in \mathbf{N}$ such that if $n \geq n_0$, then $a_n^{1/n} < q + \epsilon$. Thus, for such $n$ we have $a_n < (q+\epsilon)^n$ and

$$\sum_{n=n_0}^{\infty} a_n < \sum_{n=n_0}^{\infty} (q+\epsilon)^n < \infty$$

since $q + \epsilon < 1$ and we have a convergent geometric series on the right-hand side of the first inequality. Therefore, our series converges. The case $q > 1$ is proved in a similar way (the reader is advised to do it as an exercise). $\square$

*Remark.* It is important to note that if $q = 1$, then Cauchy's test doesn't work, i.e. the series can converge, diverge to $\infty$, diverge to $-\infty$ or diverge. For instance, notice that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}, \quad \sum_{n=1}^{\infty} \frac{1}{n}, \quad \sum_{n=1}^{\infty} \frac{-1}{n}, \quad \sum_{n=1}^{\infty} (-1)^n$$

are easy examples of such series.

EXAMPLE 6. Using Cauchy's test, we can see that the series

$$\sum_{n=1}^{\infty} \left(\frac{n-1}{2n+1}\right)^n$$

converges since

$$q = \lim_{n\to\infty} \frac{n-1}{2n+1} = \frac{1}{2} < 1.$$

Example 7. Consider the series

$$\sum_{n=1}^{\infty} n^k a^n$$

where $a > 0$. Here,

$$q = \lim_{n\to\infty} n^{k/n} a = (\lim_{n\to\infty} n^{1/n})^k a = a$$

hence the series converges for $a < 1$ and diverges to $\infty$ for $a > 1$ (it has to diverge to $\infty$ if it doesn't converge since it has only positive terms).

Theorem 5. (d'Alembert's test) *Suppose that $a_n \neq 0$ for $n \geq n_0$ and that $\lim_{n\to\infty} |a_{n+1}/a_n| = q$, by which we mean that the limit exists and is equal to $q$, allowing $q = \infty$. Then the series $\sum_{n=1}^{\infty} a_n$ converges if $q < 1$ and it does not converge if $q > 1$.*

*Remark.* The same remark as in the case of Cauchy's test applies, i.e. if $q = 1$, then the test doesn't say anything about convergence or divergence of the series (even the same examples can be used to illustrate this). Also the proof of the theorem is similar and is based on the comparison with the geometric series (the reader is advised to carry out the proof in the case of positive terms)

Example 8. Consider the series

$$\sum_{n=1}^{\infty} \frac{b^n}{n!}$$

for $b > 0$. We have

$$\frac{a_{n+1}}{a_n} = \frac{b^{n+1}}{b^n} \frac{n!}{(n+1)!} = \frac{b}{n+1} \to 0$$

and hence the series converges.