CALCULUS II

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Lecture 4

Absolute and conditional convergence

Note that not all convergence tests apply to series with both positive and negative terms. Only Cauchy's test and d'Alemebert's test are general enough in that respect. Unfortunately, it happens quite often that $q = 1$ and thus these tests don't work. Therefore, we need some convenient convergence tests for series like

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}, \quad \sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}, \quad \sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln^2 n}$$

There are two typical tools which can be used: absolute convergence (which implies convergence) and Leibniz's theorem. Let us begin with introducing the notion of absolute convergence. It can be seen that it is similar to absolute convergence of improper integrals.

Definition 1. We say that the series $\sum_{n=1}^{\infty} a_n$ *converges absolutely* if the series $\sum_{n=1}^{\infty} |a_n|$ converges.

Theorem 1. *If the series $\sum_{n=1}^{\infty} a_n$ converges absolutely then it converges and*

$$\left|\sum_{n=1}^{\infty} a_n\right| \leq \sum_{n=1}^{\infty} |a_n|$$

Example 1. The series

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln^2 n}$$

converges by Theorem 1 since it converges absolutely, which follows from the integral test (use the substitution $t = \ln x$ to show that $\int_2^{\infty} dx/(x \ln^2 x) < \infty$).

Although there are many series whose convergence can be established simply because they are absolutely convergent, the notion of absolute convergence is much stronger than that of convergence and there are plenty of examples to demonstrate this. For instance, two of the series given in the beginning of this lecture are not absolutely convergent.

Then, of course, we don't know anything about their convergence and more subtle test are needed. The most typical test is given by Leibniz's theorem.

THEOREM 2. (LEIBNIZ'S THEOREM) *Suppose that* $(b_n)_{n\geq 1}$ *is nonincreasing for* $n \geq n_0$ *and* $\lim_{n\to\infty} b_n = 0$. *Then the series*

$$\sum_{n=1}^{\infty}(-1)^n b_r$$

converges.

Remark. Notice that the assumptions imply that $b_n \geq 0$ for $n \geq n_0$ (this is sometimes stated as an explicit assumption). Of course, as in practically all other tests, what matters is the behavior of the series at infinity – that's why we have $n \geq n_0$ appearing again.

EXAMPLE 2. This test allows us to deal with the two series

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n}, \quad \sum_{n=2}^{\infty}\frac{(-1)^n}{n\ln n}$$

which are not absolutely convergent. The first one is the easiest application of Theorem 2. It is enough to see that the sequence $b_n = 1/n \downarrow 0$, i.e. decreases and has limit equal to 0. The second series can be treated in a similar manner. Again, if we take $b_n = 1/(n\ln n) \downarrow 0$ and apply Theorem 2, we get convergence of our series.

EXAMPLE 3. Let us consider more sophisticated examples. For instance, let us study convergence of the series

$$\sum_{n=1}^{\infty}\frac{(-1)^n\ln n}{n}.$$

Although each term of the series can be written as $(-1)^n b_n$, where b_n is positive, this is far from enough. We need to show that $\lim_{n\to\infty} b_n = 0$ *and* that b_n is non-increasing starting from some $n = n_0$. To evaluate the limit we take the function $f(x) = \ln x/x$ and evaluate the limit

$$\lim_{x\to\infty}\frac{\ln x}{x} = \lim_{x\to\infty} 1/x = 0$$

using de L'Hospital's rule. Even that is not enough though since we still haven't shown that the sequence b_n is non-increasing. Again, we will look at the function $f(x)$. Of course, if the function $f(x)$ is non-increasing for $x \geq x_0$, then the sequence b_n is non-increasing for $n \geq n_0$ for some n_0 and it is much easier to show monotonicity of a function using derivatives than monotonicity of a sequence using the definition). Thus,

$$f'(x) = \frac{1-\ln x}{x^2}$$

and hence, if $x > e$ then f is decreasing. Thus, if $n \geq n_0 = 3$, then $b_n = \ln n/n \downarrow 0$. Therefore, we can use Theorem 2 to conclude that our series converges.