Hilbert Space Approach to Limit Distributions of Random Matrices and the Triangular Operator

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Introduction

Free probability and its generalizations give several ways to treat various families of random matrices:

1. *freeness* (free probability, scalar-valued state, free Fock space)
2. *matricial freeness* (matricially free probability, family of scalar-valued states, matricially free Fock space)
3. *freeness with amalgamation* (operator-valued free probability, conditional expectation, free Fock space over a Hilbert bimodule)

We will use matricial freeness, which can be viewed as a 'scalar-valued approach to freeness with amalgamation'.

Introduction

My talk refers to the asymptotics of independent random matrices:

- Introduction
- Gaussian random matrices
- Direct integrals
- Triangular operators
- Products of Wishart type

Realizations of limit distributions are on Hilbert spaces rather than on Hilbert modules (Shlyakhtenko).

The general scheme is the following:

1. Consider a family of independent random matrices

$$\mathcal{Y} = \{Y(u,n) : u \in \mathcal{U}, n \in \mathbb{N}\}$$

2. Determine limit *-moments of their blocks $S_{i,j}(u,n)$
3. Find a Hilbert space realization of the limit *-moments
4. Describe their combinatorics
5. Colored labeled noncrossing partitions
6. Here: i, j - colors, u - labels

Normalized partial traces

1. Decompose the set $[n] := \{1, \ldots, n\}$ into disjoint intervals

$$[n] = N_1 \cup \ldots \cup N_r$$

and set $n_j = |N_j|$ (n suppressed).

2. Use normalized partial traces

$$\tau_j = \mathbb{E} \circ \mathrm{Tr}_j$$

where

$$\mathrm{Tr}_j(A) = \frac{n}{n_j} \mathrm{Tr}(D_j(n) A D_j(n))$$

and $D_j(n)$ is the $n \times n$ partial unit matrix embedded in $M_n(\mathbb{C})$ at the places indexed by the set N_j.

I. Gaussian random matrices

Decomposition into blocks

Assume that complex matrices $Y(u,n)$ have block-identical variances $v_{i,j}(u)$ (*i.b.i.d.*) and decompose them:

$$Y(u,n) = \sum_{i,j=1}^{r} S_{i,j}(u,n),$$

where $S_{i,j}(u,n) = D_i(n)Y(u,n)D_j(n)$.

Definition

The *matricially free Fock space of tracial type* over an array $(\mathcal{H}_{i,j})$ of Hilbert spaces is the direct sum

$$\mathcal{M} = \bigoplus_{j=1}^{r} \mathcal{M}_j,$$

where

$$\mathcal{M}_j = \mathbb{C}\Omega_j \oplus \bigoplus_{m=1}^{\infty} \bigoplus_{\substack{j_1,\ldots,j_m \\ u_1,\ldots,u_m}} \mathcal{H}_{j_1,j_2}(u_1) \otimes \mathcal{H}_{j_2,j_3}(u_2) \otimes \ldots \otimes \mathcal{H}_{j_m,j}(u_m)$$

with the canonical inner product. Let Ψ_j be the state associated with Ω_j and let P_j be the associated orthogonal projection.

Definition

The *matricially free creation operators* are operators on $\mathcal{M}$ defined by

$$\begin{aligned}\wp_{i,j}(u)\Omega_j &= \sqrt{b_{i,j}(u)}e_{i,j}(u)\\ \wp_{i,j}(u)(e_{j,k}(s)) &= \sqrt{b_{i,j}(u)}e_{i,j}(u)\otimes e_{j,k}(s)\\ \wp_{i,j}(u)(e_{j,k}(s)\otimes w) &= \sqrt{b_{i,j}(u)}e_{i,j}(u)\otimes e_{j,k}(s)\otimes w\end{aligned}$$

for any $i,j,k\in[r]$ and $u,s\in\mathcal{U}$, where $e_{j,k}(s)\otimes w$ is a basis vector and

$$b_{i,j}(u)=d_i v_{i,j}(u)$$

where $d_j=\lim_{n\to\infty} n_j/n$ (asymptotic dimensions). The actions onto the remaining basis vectors give zero.

Theorem ('16)

Let $\mathcal{Y}$ be the family of independent Gaussian random matrices with block-identical variances $v_{i,j}(u)$. Then

$$\lim_{n\to\infty} \tau_j(S_{i_1,j_1}^{\epsilon_1}(u_1,n)\cdots S_{i_m,j_m}^{\epsilon_m}(u_m,n)) = \Psi_j(\zeta_{i_1,j_1}^{\epsilon_1}(u_1)\cdots\zeta_{i_m,j_m}^{\epsilon_m}(u_m))$$

where operators

$$\zeta_{i,j}(u) = \wp_{i,j}(u') + \wp_{j,i}^{*}(u'')$$

where $u' \neq u''$ are two 'copies' of u, form *matricial circular systems* and $\epsilon_1,\ldots,\epsilon_m \in \{1,*\}$, $u_1,\ldots,u_m \in \mathcal{U}$.

Remark

In the i.i.d. case, under partial traces (and thus under the expectation of normalized trace τ) it holds that

$$\lim_{n\to\infty} D_j(n) = P_j$$
$$\lim_{n\to\infty} Y(u,n) = \ell(u') + \ell^*(u'')$$

where $\ell(u'), \ell(u'')$ are free creation operators realized on $\mathcal{M}$. This is the original Voiculescu's case (1991), in his notation we get circular operators of the form $\ell_{2j-1} + \ell^*_{2j}$, but the Fock space is different. Roughly speaking, in the general case

$$\wp_{i,j}(u) = P_i L(u) P_j$$

where $L(u)$ is the creation operator living in the free Fock module.

Sums of operators

In the present context, we are looking for continuous analogs of sums of matricially free creation operators

$$\wp(u) = \sum_{i,j=1}^{r} \wp_{i,j}(u),$$

of their adjoints, and of the sums of matricial circular operators

$$\zeta(u) = \sum_{i,j=1}^{r} \zeta_{i,j}(u)$$

In particular, if $b_{p,q}(u) = d_p$ for any p, q, u, we obtain decompositions of the canonical free creation operators and free circular operators.

Measure space

For $I = [0,1]$, let

$$\Gamma = \bigoplus_{n=0}^{\infty} \Gamma_n$$

be the direct sum of measure spaces, where $\Gamma_n = I^{n+1}$ is equipped with the Lebesgue measure denoted $d\gamma_n$, and let $d\gamma$ be the corresponding direct sum of measures on the set Γ.

II. Direct integrals

Definition

By the *continuous matricially free Fock space* we understand the direct integral of Hilbert spaces of the form

$$\mathcal{H} = \int_{\Gamma}^{\oplus} \mathcal{H}(\gamma) d\gamma,$$

1. if $\gamma = x \in \Gamma_0 = I$, then
$$\mathcal{H}(\gamma) = \mathbb{C}\Omega(x),$$
where $\Omega(x)$ is a unit vector,
2. if $\gamma = (x_1, x_2, \ldots, x_{n+1}) \in \Gamma_n$ and $n \in \mathbb{N}$, then
$$\mathcal{H}(\gamma) = \mathcal{H}(x_1, x_2) \otimes \mathcal{H}(x_2, x_3) \otimes \ldots \otimes \mathcal{H}(x_n, x_{n+1}),$$
where each $\mathcal{H}(x, y)$ is a separable Hilbert space.

Inner product

Each $\mathcal{H}(\gamma)$ is equipped with the canonical inner product, the canonical inner product in $\mathcal{H}$ is then given by

$$\langle F, G\rangle = \int_{\Gamma} \langle F(\gamma), G(\gamma)\rangle \, d\gamma,$$

where $F = \int_{\Gamma}^{\oplus} F(\gamma)d\gamma, G = \int_{\Gamma}^{\oplus} G(\gamma)d\gamma \in \mathcal{H}$ are square integrable fields, with $F(\gamma), G(\gamma) \in \mathcal{H}(\gamma)$.

Remarks

1. The direct integral
$$\mathcal{H}_0 := \int_I^{\oplus} \mathcal{H}(x)dx \cong L^2(I)$$
is the *vacuum space*.
2. The corresponding state is
$$\varphi = \int_I^{\oplus} \varphi(\gamma)d\gamma$$
3. If $\mathcal{H}(x,y) \cong \mathcal{G}$ for any $(x,y) \in \Gamma_1$, where $\mathcal{G}$ is a separable Hilbert space (with an orthonormal basis indexed by $\mathcal{U}$), then
$$\mathcal{H}_n := \int_{\Gamma_n}^{\oplus} \mathcal{H}(\gamma)d\gamma_n \cong L^2(\Gamma_n, \mathcal{G}^{\otimes n}),$$

Proposition

Fields $F = \int_{\Gamma}^{\oplus} F(\gamma)d\gamma, G = \int_{\Gamma}^{\oplus} G(\gamma)d\gamma \in \mathcal{H}$ have direct sum decompositions

$$F = \sum_{n=0}^{\infty} F_n \text{ and } G = \sum_{n=0}^{\infty} G_n,$$

where $F_n, G_n \in \int_{\Gamma_n} \mathcal{H}(\gamma)d\gamma$. Under the above isomorphism assumptions, $F_0, G_0 \in L^2(I)$ and $F_n, G_n \in L^2(\Gamma_n, \mathcal{G}^{\otimes n})$ for $n \geqslant 1$. It is enough to consider these to be of the form

$$\begin{aligned} F_n(\gamma) &= f_1(x_1, x_2) \otimes \ldots \otimes f_n(x_n, x_{n+1}), \\ G_n(\gamma) &= g_1(x_1, x_2) \otimes \ldots \otimes g_n(x_n, x_{n+1}), \end{aligned}$$

for $\gamma = (x_1, \ldots, x_{n+1})$ and $n \geqslant 1$.

Proposition

The canonical inner product in $\mathcal{H}$ decomposes as

$$\langle F, G\rangle = \sum_{n=0}^{\infty} \int_{\Gamma_n} \langle F_n(\gamma), G_n(\gamma)\rangle \, d\gamma_n$$

for any $F, G \in \mathcal{H}$, and an analogous equation holds for squared norms

$$\|F\|^2 = \sum_{n=0}^{\infty} \int_{\Gamma_n} \|F_n(\gamma)\|^2 \, d\gamma_n$$

Canonical operators on $\mathcal{H}$

Let $f \in L^{\infty}(\Gamma_1, \mathcal{G})$ and $h \in L^{\infty}(\Gamma_1)$. The canonical operators on $\mathcal{H}$ are:

- $\wp(f) : \mathcal{H}_n \to \mathcal{H}_{n+1}$ (*creation operators*)
- $M(h) : \mathcal{H}_n \to \mathcal{H}_n$ (*multiplication operators*)
- $\wp^*(f) : \mathcal{H}_{n+1} \to \mathcal{H}_n$ (*annihilation operators*)

where $n \in \mathbb{N} \cup \{0\}$.

II. Direct integrals

Definition

For given $f \in L^{\infty}(\Gamma_1, \mathcal{G})$, define creation operators by

$$\wp(f)\left(\int_I^{\oplus} F_0(x_1)dx_1\right) = \int_{\Gamma_1}^{\oplus} f(x, x_1)F_0(x_1)dxdx_1$$

for any $F_0 \in L^2(I)$, and

$$\wp(f)\left(\int_{\Gamma_n}^{\oplus} F_n(x_1, \ldots, x_{n+1})dx_1 \ldots dx_{n+1}\right)$$

$$= \int_{\Gamma_{n+1}}^{\oplus} f(x, x_1) \otimes F_n(x_1, \ldots, x_{n+1})dxdx_1 \ldots dx_{n+1}$$

for any $F_n \in L^2(\Gamma_n, \mathcal{G}^{\otimes n})$, where $n \in \mathbb{N}$. If $f = g \otimes e(u)$, where $e(u)$ is a basis unit vector of $\mathcal{G}$, under the identification $L^{\infty}(\Gamma_1, \mathcal{G}) \cong L^{\infty}(\Gamma_1) \otimes \mathcal{G}$, we write $\wp(g, u)$ instead of $\wp(f)$.

Definition

For $h \in L^{\infty}(I)$, define bounded linear operators

$$M(h,\gamma) : \mathcal{H}(\gamma) \to \mathcal{H}(\gamma)$$

for any $\gamma = (x_1, \ldots, x_{n+1}) \in \Gamma_n$ and $n \geqslant 0$ by

$$M(h,\gamma)F_n(\gamma) = h(x_1)F_n(\gamma),$$

and the associated decomposable operator in the direct integral form

$$M(h) := \int_{\Gamma}^{\oplus} M(h,\gamma)d\gamma,$$

which is a bounded linear operator on $\mathcal{H}$.

II. Direct integrals

Notation

To define the annihilation operators, denote

$$
\begin{aligned}
F_n(\gamma) &= f_1(x_1, x_2) \otimes \ldots \otimes f_n(x_n, x_{n+1}), \\
F_{n-1}(\gamma') &= f_2(x_2, x_3) \otimes \ldots \otimes f_n(x_n, x_{n+1}),
\end{aligned}
$$

where $\gamma = (x_1, \ldots, x_{n+1}) \in \Gamma_n$, $\gamma' = (x_2, \ldots, x_{n+1}) \in \Gamma_{n-1}$ and each $f_i \in \mathcal{G}$.

Proposition

The adjoints of the operators $\wp(f)$ are given by

$$\begin{aligned}\wp^*(f)\int_I^{\oplus} F_0(\gamma)d\gamma_0 &= 0\\ \wp^*(f)\int_{\Gamma_n}^{\oplus} F_n(\gamma)d\gamma_n &= \int_{\Gamma_{n-1}}^{\oplus} M(h,\gamma')F_{n-1}(\gamma')d\gamma'_{n-1}\end{aligned}$$

where

$$h(x) = \int_0^1 \langle f_1(y,x), f(y,x)\rangle dy,$$

and $\langle .,.\rangle$ is the canonical inner product in $\mathcal{G}$.

Corollary

For any $f, g \in L^{\infty}(\Gamma_1, \mathcal{G})$, it holds that

$$\wp^{*}(f)\wp(g) = M(h),$$

where

$$h(x) = \int_0^1 \langle g(y,x), f(y,x)\rangle dy,$$

II. Direct integrals

Circular operators

1. If $f(x,y) = \widetilde{f}(x)$ and $g(x,y) = \widetilde{g}(x)$, then

$$h(y) = \int_0^1 \langle \widetilde{g}(x), \widetilde{f}(x) \rangle dx,$$

   and thus

$$\wp^*(f)\wp(g) = \langle g, f \rangle = \langle \widetilde{g}, \widetilde{f} \rangle,$$

   and the operators $\wp(g), \wp^*(f)$ reduce to free creation and annihilation operators, respectively.

2. If $f = \chi_{\Gamma_1} \otimes e(u')$ and $g = \chi_{\Gamma_1} \otimes e(u'')$, with $e(u') \perp e(u'')$, then we get free circular operators by taking

$$\zeta(u) = \wp(f) + \wp^*(g)$$

Triangular operators

If $f = \chi_{\Delta} \otimes e(u_1)$, $g = \chi_{\Delta} \otimes e(u_2)$, $\Delta = \{(x,y) : 0 \leqslant x < y \leqslant 1\}$, and $e(u_1), e(u_2)$ are orthonormal basis vectors in $\mathcal{G}$, then

$$h(y) = \delta_{u_1,u_2} \int_0^y dx = \delta_{u_1,u_2} y,$$

and thus

$$\wp^*(f)\wp(g) = \delta_{u_1,u_2} M(id),$$

which corresponds to the case when we deal with strictly upper triangular Gaussian random matrices and the operatorial limit is the triangular operator of Dykema and Haagerup (2004).

Theorem ('18)

Let $\{Y(u,n,r) : u \in \mathcal{U}, r \in \mathbb{N}\}$ be a family of independent $n \times n$ random matrices for any $n \in \mathbb{N}$, such that

1. each $Y(u,n,r)$ consists of r^2 blocks of equal size with i.b.i.d. complex Gaussian entries,
2. the sequence of simple functions (b_r) converges to g in $L^{\infty}(\Gamma_1)$ as $r \to \infty$.

Then

$$\lim_{r\to\infty}\lim_{n\to\infty} \tau(n)(Y^{\epsilon_1}(u_1,n,r)\ldots Y^{\epsilon_m}(u_m,n,r))$$
$$= \varphi(\zeta^{\epsilon_1}(g,u_1)\ldots\zeta^{\epsilon_m}(g,u_m)),$$

where $\zeta(g,u_j) = \wp(g,u_j') + \wp^*(g^t,u_j'')$, with $g^t(x,y) = g(y,x)$.

Corollary

Let $\{Y(u,n) : u \in \mathcal{U}\}$ be a family of independent strictly upper triangular Gaussian random matrices for any $n \in \mathbb{N}$. Then

$$\lim_{n\to\infty} \tau(n)(Y^{\epsilon_1}(u_1,n)\ldots Y^{\epsilon_m}(u_m,n)) = \varphi(\zeta^{\epsilon_1}(u_1)\ldots\zeta^{\epsilon_m}(u_m))$$

for any $\epsilon_1,\ldots,\epsilon_m \in \{1,*\}$ and $u_1,\ldots,u_m \in \mathcal{U}\}$, where $\zeta(u_j) = \zeta(\chi_\Delta, u_j)$, $j \in [m]$.

Definition

Let $\mathcal{NC}^2((\epsilon_1, u_1), \ldots, (\epsilon_m, u_m))$ be the set of noncrossing pair partitions of $[m]$, such that $u_i = u_j$ and $\epsilon_i \neq \epsilon_j$ whenever $\{i, j\}$ is a block.

Remark

The combinatorics of *-moments of operators $\zeta(f)$, where $f = \chi_\Delta \otimes e(u)$ is based on coloring the blocks of $\pi \in \mathcal{NC}^2((\epsilon_1, u_1), \ldots, (\epsilon_m, u_m))$, where $m = 2n$, and the imaginary block with $n + 1$ continuous colors form $[0, 1]$: $x_1, \ldots, x_{n+1}$, assigned to $V_1, \ldots, V_{n+1}$, respectively.

III. Triangular operators

Definition

Let $o(V)$ be the nearest outer block of V. Associate a region $V^*(\pi) \subset \Gamma_n$ to each π:

$$V^*(\pi) = \{x : \; x_j < x_{o(j)} \text{ if } V_j \in B'(\pi) \; \wedge \; x_j > x_{o(j)} \text{ if } V_j \in B''(\pi)\},$$

where $B'(\pi)$ and $B''(\pi)$ are families of blocks of π whose left legs are starred and unstarred, respectively.

III. Triangular operators

Theorem

The non-vanishing mixed *-moments of the free triangular operators $T(u_j) = \zeta(\chi_\Delta, u_j)$ in the state φ take the form

$$\varphi(T^{\epsilon_1}(u_1)\ldots T^{\epsilon_m}(u_m)) = \sum_{\pi \in \mathcal{NC}^2((\epsilon_1, u_1), \ldots, (\epsilon_m, u_m))} Vol^*(\pi),$$

where $m = 2n$ and $Vol^*(\pi)$ is the volume of the region $V^*(\pi)$.

Proposition

The volume $V^*(\pi)$ is equal to the number of simplices defined by the relations between the colors of blocks of π multiplied by $1/(n+1)!$ (the volume of a standard $n+1$-dimensional simplex).

III. Bijection

Lemma

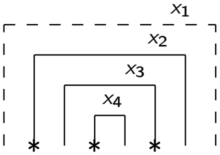
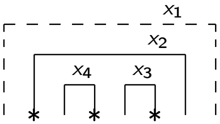
There is a natural bijection

$$\mathcal{A}_m \cong \mathcal{ACNC}^2(2m)$$

where:
$\mathcal{A}_m$ - alternating ordered rooted trees of type A on $m+1$ vertices,
$\mathcal{ACNC}^2(2m)$ - alternating colored noncrossing pair partitions of type A of $[2m]$.

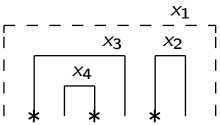
Type A means that: the label of the root is smaller than the labels of its children and that the color of the imaginary block is smaller than the colors of its neighboring inner blocks.

III. Noncrossing colored pair partitions

partition	*relations*	*simplices*
π_1 : 	$x_2 < x_1$ $x_2 < x_3$ $x_4 < x_3$	$x_4 < x_2 < x_3 < x_1$ $x_2 < x_4 < x_3 < x_1$ $x_2 < x_4 < x_1 < x_3$ $x_4 < x_2 < x_1 < x_3$ $x_2 < x_1 < x_4 < x_3$
π_2 : 	$x_2 < x_1$ $x_2 < x_4$ $x_2 < x_3$	$x_2 < x_3 < x_4 < x_1$ $x_2 < x_4 < x_3 < x_1$ $x_2 < x_4 < x_1 < x_3$ $x_2 < x_3 < x_1 < x_4$ $x_2 < x_1 < x_3 < x_4$ $x_2 < x_1 < x_4 < x_3$

III. Noncrossing colored pair partitions

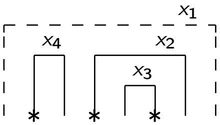
π_3 :



$x_2 < x_1$
$x_3 < x_4$
$x_3 < x_1$

$x_2 < x_3 < x_4 < x_1$
$x_3 < x_2 < x_4 < x_1$
$x_3 < x_4 < x_2 < x_1$
$x_3 < x_2 < x_1 < x_4$
$x_2 < x_3 < x_1 < x_4$

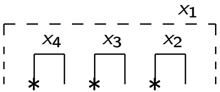
π_4 :



$x_2 < x_1$
$x_4 < x_1$
$x_2 < x_3$

$x_4 < x_2 < x_3 < x_1$
$x_2 < x_4 < x_3 < x_1$
$x_2 < x_3 < x_4 < x_1$
$x_4 < x_2 < x_1 < x_3$
$x_2 < x_4 < x_1 < x_3$

π_5 :



$x_2 < x_1$
$x_3 < x_1$
$x_4 < x_1$

$x_4 < x_3 < x_2 < x_1$
$x_3 < x_4 < x_2 < x_1$
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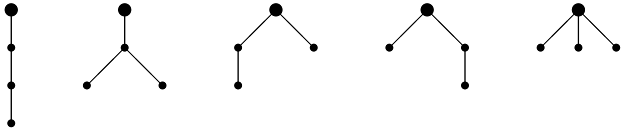
$x_2 < x_4 < x_3 < x_1$
$x_3 < x_2 < x_4 < x_1$
$x_2 < x_3 < x_4 < x_1$

III. Ordered rooted trees on 4 vertices



T_1 T_2 T_3 T_4 T_5

III. Alternating ordered rooted trees of type A

Enumeration

There are $27 = 3^3$ alternating ordered rooted trees of type A on 4 vertices.

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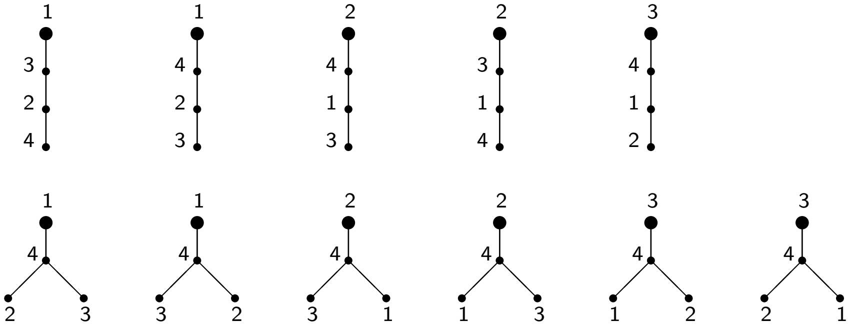
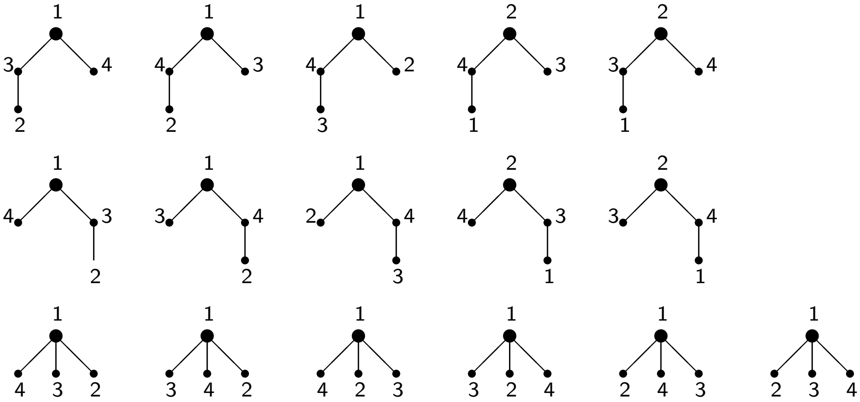


III. Alternating ordered rooted trees of type A



III. Alternating ordered rooted trees

Theorem (Chauve, Dulucq and Rechnitzer 2001)

The cardinality of the set of alternating ordered rooted trees of type A on $m+1$ vertices is

$$|\mathcal{A}_m| = m^m$$

for any natural m. Thus, the number of all alternating ordered rooted trees is $2m^m$.

Corollary

Using bijection 3, we obtain a bijective proof of the known formula

$$\varphi((TT^*)^m) = \frac{m^m}{(m+1)!}$$

the result proved by Dykema and Haagerup (2004), generalized by Śniady (2003) to moments of $T^k(T^*)^k$.

IV. Products of Wishart type

Assumptions

Consider the family $\mathcal{Y} = \{Y(u,n) : u \in \mathcal{U}\}$ of independent Hermitian random matrices for each $n \in \mathbb{N}$, such that

1. $\mathcal{Y}$ is asymptotically free,
2. $\mathcal{Y}$ is asymptotically free against the family of constant diagonal matrices,
3. the norms of $Y(u,n)$ are uniformly bounded almost surely.

IV. Products of Wishart type

Three cases

We study matrices of Wishart type BB^*, where

$$B = X_1 X_2 \ldots X_p$$

in three situations:

1. blocks $X_j = S_{j,j+1}(u, n)$ are taken from one Gaussian random matrix $Y(u, n)$
2. blocks $X_j = S_{j,j+1}(u_j, n)$ are taken from independent matrices from class $\mathcal{Y}$
3. blocks $X_j = S_{j,j+1}(u, n)$ are taken from one arbitrary matrix $Y(u, n)$ from class $\mathcal{Y}$.

IV. Multivariate Fuss-Narayana polynomials

Definition

By *multivariate Fuss-Narayana polynomials* we understand polynomials of the form

$$P_m(d_1, \ldots, d_{p+1}) = \sum_{j_1+\ldots+j_{p+1}=pm+1} \frac{1}{m}\binom{m}{j_1}\ldots\binom{m}{j_{p+1}} d_1^{j_1}\ldots d_{p+1}^{j_{p+1}}.$$

where $m \in \mathbb{N}$ and the summation runs over nonnegative integers. These polynomials generalize Narayana polynomials (reproduced for $p = 1$ under different normalization).

Theorem 1 (R.L. & R. Sałapata '12, Müller '02)

If $B = X_1 X_2 \ldots X_p$, where $X_1, \ldots, X_p$ are independent standard GRM of sizes $n_1 \times n_2, \ldots, n_p \times n_{p+1}$, then

$$\lim_{n \to \infty} \tau_1((BB^*)^m) = d_1^{-1} P_m(d_1, \ldots, d_{p+1})$$

IV. Generalized multivariate Fuss-Narayana polynomials

Definition

By *generalized multivariate Fuss–Narayana polynomials* we understand polynomials of the form

$$P_{m,r}(d_1,\ldots,d_{p+1}) = \sum_{j_1+\ldots+j_{p+1}=mp+r} \frac{1}{k}\binom{m}{j_1}\cdots\binom{m}{j_{p+1}} d_1^{j_1}\ldots d_{p+1}^{j_{p+1}}.$$

where $m, r \in \mathbb{N}$ and summation runs over nonnegative integers.

IV. Moment generating function

Lemma

The moment generating function ψ_μ of the asymptotic distribution μ of BB^* under τ_1 is the unique solution of the equation

$$d_1\psi_\mu = R_{\widetilde{\nu}}\big(z(d_1\psi_\mu + d_1)(d_1\psi_\mu + d_2)\ldots(d_1\psi_\mu + d_{p+1})\big)$$

where $\widetilde{\nu} = \widetilde{\nu}_1 \boxtimes \ldots \boxtimes \widetilde{\nu}_p$ and $\widetilde{\nu}_j$ is defined by the even free cumulants of ν_j for $j \in [p]$.

Theorem 2 (R.L. & Rafał Sałapata '16)

In the general case of Wishart type products of independent matrices,

$$\lim_{n\to\infty}\tau_1((BB^*)^m) = d_1^{-1}\sum_{r=1}^{m} P_{m,r}(d_1,\ldots,d_{p+1})\,T_{m,r}(t_1,\ldots,t_r)$$

where $T_{m,r}$ depends on the coefficients of the T-transform of $\widetilde{\nu}$ (the reciprocal S-transform $T_\mu(z) = 1/S_\mu(z)$, used by Dykema and Nica in some papers).

Definition

For given natural numbers m and p, we will say $\pi \in \mathrm{NC}(2pm)$ is *adapted* to the word

$$W^m = (12\ldots pp^*\ldots 2^*1^*)^m$$

if and only if

1. each block V of π contains numbers associated with the same labels,
2. each block V of π contains the same number of each letter and its starred counterpart.

This family of partitions will be denoted by $\mathrm{NC}^e(W^m)$.

Lemma

In the general case of Wishart type products,

$$M_m = \sum_{\pi \in \mathrm{NC}^e(W^m)} w(\pi),$$

where

$$w(\pi) = \prod_{\text{blocks } V} w(V),$$

and $w(V) = d(V) r_{|V|}(u_V)$, where u_V is the labelling associated with V and $d(V)$ is the asymptotic dimension factor assigned to block V and $r_{|V|}(u_V)$ is the free cumulant assigned to V.

Lemma

Let $W = 12\dots pp^*\dots 2^*1^*$ and let $\widetilde{W} = 12\dots 2p(2p)^*\dots 2^*1^*$. Then there is a bijection

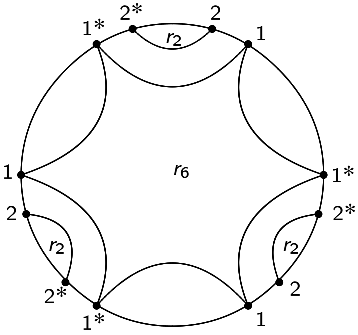
$$\alpha : \mathrm{NC}^e(W^m) \to \mathrm{NC}^2(\widetilde{W}^m),$$

where $\mathrm{NC}^2(\widetilde{W}^m)$ is the set of pair partitions (truly) adapted to $\widetilde{W}$, which means that it consists of blocks of the form $\{j, j^*\}$.

IV. Bijection $\mathrm{NC}^e(W^3) \cong \mathrm{NC}^2(\widetilde{W}^3)$

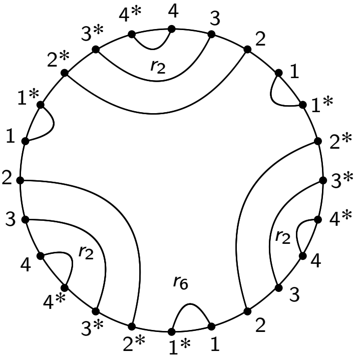
Assume that all labels are different, as in Theorem 2.

π



W=122*1*

$\alpha(\pi)$



$\widetilde{W}$=12344*3*2*1*

Corollary

Bijection α leads to the following Gaussianization:

$$M_m = \sum_{\sigma \in \mathrm{NC}^2(\widetilde{W}^m)} \widetilde{w}(\sigma),$$

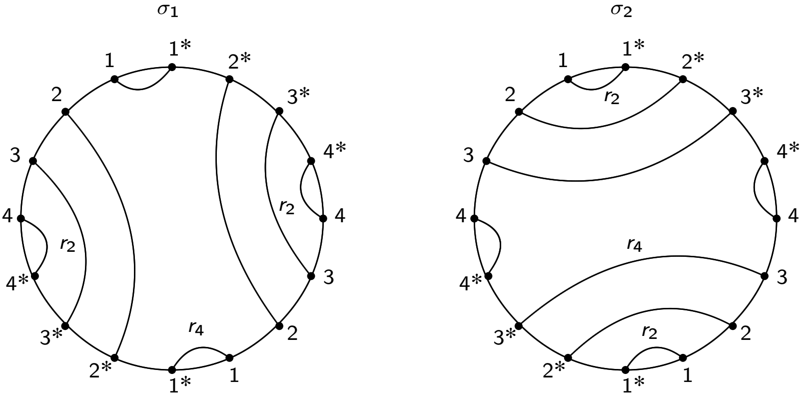
where

$$\widetilde{w}(\sigma) = \prod_{\text{blocks } V} \widetilde{w}(V),$$

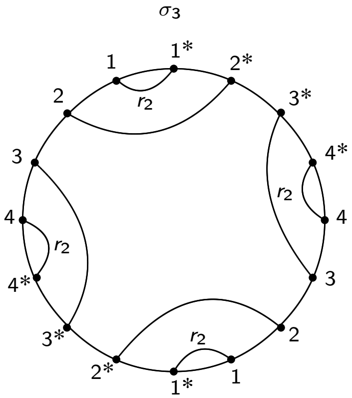
with weights induced from the weights on blocks of $\pi = \alpha^{-1}(\sigma)$.

IV. $\mathrm{NC}^2(\widetilde{W}^2)$ for $\widetilde{W} = 12344^*3^*2^*1^*$

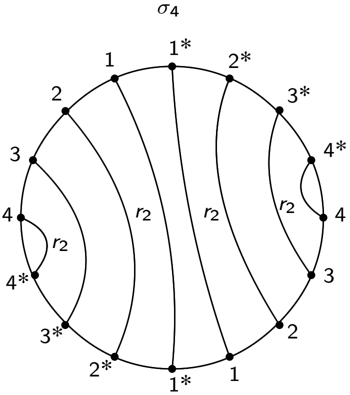
There are $C_2(4) = 5$ partitions in this set.

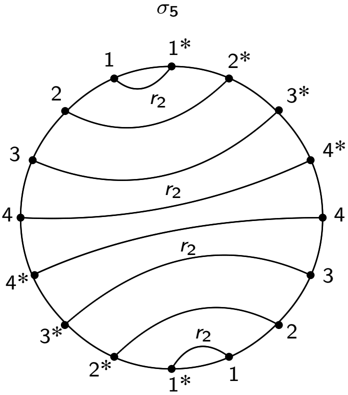


IV. $\mathrm{NC}^2(\widetilde{W}^2)$ for $\widetilde{W} = 12344^*3^*2^*1^*$



IV. $\mathrm{NC}^2(\widetilde{W}^2)$ for $\widetilde{W} = 12344^*3^*2^*1^*$





Thank you for your attention!