Limit distributions of random matrices

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Free probability and random matrices

Significance of free probability to random matrix theory lies in the fundamental observation that random matrices which are independent in the classical sense also tend to be independent in the free probability sense in the large $n \to \infty$ limit.

Many tedious computations in random matrix theory, particularly those of an algebraic or enumerative combinatorial nature, can be done more quickly and systematically by using the framework of free probability.

Terrence Tao

My motivations

My motivations:

- *unify random matrix models*
- *describe limit distributions of random matrices*
- *construct new random matrix models*

Different versions of free probability

Free probability and its generalizations give a way to treat various families of independent random matrices:

1. *freeness (scalar-valued states, free probability)*
2. *matricial freeness (families of scalar-valued states, matricially free probability)*
3. *freeness with amalgamation (operator-valued states, operator-valued free probability)*

Fundamental results

1. If $Y(s,n)$ is a standard complex HGRM, it converges in moments to a semicircular operator

$$\lim_{n\to\infty} Y(s,n) \to \omega_s$$

under $\tau(n) = \mathbb{E} \circ \mathrm{Tr}(n)$ (Wigner).

2. If $Y(s,n)$ is a standard complex GRM, it converges in *-moments to a circular operator

$$\lim_{n\to\infty} Y(s,n) \to \eta_s$$

under $\tau(n) = \mathbb{E} \circ \mathrm{Tr}(n)$ (Ginibre).

Voiculescu's asymptotic freeness and generalizations

1. Complex independent HGRM converge to a free semicircular family
$$\{Y(s,n) : s \in S\} \to \{\omega_s : s \in S\}$$
2. Complex independent GRM converge to a *-free circular family
$$\{Y(s,n) : s \in S\} \to \{\eta_s : s \in S\}$$
3. Generalizations (Dykema, Schlyakhtenko, Hiai-Petz, Benaych-Georges and others)

In the case of two independent random matrices, we get convergence

$$\begin{aligned} Y(1,n) &\rightarrow \omega_1 = \ell_1 + \ell_1^* \\ Y(2,n) &\rightarrow \omega_2 = \ell_2 + \ell_2^* \end{aligned}$$

under $\tau(n)$, where

$$\ell_1\Omega = e_1, \;\; \ell_2\Omega = e_2,$$

$$\ell_1 e_1^{\otimes n} = e_1^{\otimes(n+1)}, \;\; \ell_2 e_2^{\otimes n} = e_2^{\otimes(n+1)},$$

$$\ell_2 e_1^{\otimes n} = e_2 \otimes e_1^{\otimes n}, \;\; \ell_1 e_2^{\otimes n} = e_1 \otimes e_2^{\otimes n},$$

etc., are isometries on the free Fock space (free creation operators).

Typical computations

Typical computations under the complete trace:

$$
\begin{aligned}
\tau(n)(Y(1,n)Y(2,n)Y(2,n)Y(1,n)) &\to \varphi(\omega_1\omega_2\omega_2\omega_1)\\
&= \varphi(\ell_1^*\ell_2^*\ell_2\ell_1)\\
&= 1\\
\tau(n)(Y(2,n)Y(2,n)Y(1,n)Y(1,n)) &\to \varphi(\omega_2\omega_2\omega_1\omega_1)\\
&= \varphi(\ell_2^*\ell_2\ell_1^*\ell_1)\\
&= 1\\
\tau(n)(Y(2,n)Y(1,n)Y(2,n)Y(1,n)) &\to \varphi(\omega_2\omega_1\omega_2\omega_1)\\
&= 0
\end{aligned}
$$

Decomposition method

By asymptotic freeness, large HGRM are free Gaussians, so it is natural to

1. decompose them into blocks

$$Y(s,n) = \sum_{i,j} T_{i,j}(s,n)$$

2. decompose free Gaussians

$$\omega(s) = \sum_{i,j} \omega_{i,j}(s)$$

3. look for a concept of independence for the summands
4. reduce all computations to properties of these summands

Block decomposition

1. decompose the unit matrices into submatrices

$$I(n) = D_1 + \ldots + D_r$$

   where $D_j = D_j(n)$ are $0-1$ diagonal matrices,

2. use normalized partial traces

$$\tau_j(n) = \mathbb{E} \circ \mathrm{Tr}_j(n)$$

   where

$$\mathrm{Tr}_j(n)(A) = \frac{n}{n_j}\mathrm{Tr}(n)(D_j A D_j)$$

   and n_j is the number of 1s in D_j.

Block decomposition

1. decompose random matrices $Y(u,n)$ into blocks

$$S_{i,j}(s,n) = D_i Y(s,n) D_j$$

2. form symmetric blocks

$$T_{i,j}(s) = \begin{cases} S_{j,j}(s) & \text{if } i = j \\ S_{i,j}(s) + S_{j,i}(s) & \text{if } i \neq j \end{cases}$$

Matricially free Fock space

In order to find an operatorial realization of limit distributions, we need a new concept of Fock space.

Definition

By the matricially free Fock space we understand

$$\mathcal{M} = \bigoplus_{j=1}^{r} \mathcal{M}_j,$$

where each summand is of the form

$$\mathcal{M}_j = \mathbb{C}\Omega_j \oplus \bigoplus_{m=1}^{\infty} \bigoplus_{\substack{j_1,\ldots,j_m \\ s_1,\ldots,s_m}} \mathcal{H}_{j_1,j_2}(s_1) \otimes \ldots \otimes \mathcal{H}_{j_m,j}(s_m),$$

The state associated with Ω_j is denoted Ψ_j.

Creation operators

Definition

Define matricially free creation operators on $\mathcal{M}$ as partial isometries with the action onto basis vectors

$$\begin{aligned}
\wp_{i,j}(s)\Omega_j &= e_{i,j}(s) \\
\wp_{i,j}(s)(e_{j,k}(u)) &= e_{i,j}(s) \otimes e_{j,k}(u) \\
\wp_{i,j}(s)(e_{j,k}(u) \otimes w) &= e_{i,j}(s) \otimes e_{j,k}(u) \otimes w
\end{aligned}$$

for any $i, j, k \in [r]$ and $s, u \in \mathcal{U}$, where $e_{j,k}(u) \otimes w$ is a basis vector. Their actions onto the remaining basis vectors give zero.

Toeplitz-Cuntz-Krieger algebras

One square matrix of creation operators $(\wp_{i,j})$ gives an array of partial isometries satisfying relations

$$\sum_{j=1}^{r} \wp_{i,j}\wp_{i,j}^{*} = \wp_{k,i}^{*}\wp_{k,i} - \wp_{i} \text{ for any } k$$

$$\sum_{j=1}^{r} \wp_{k,j}^{*}\wp_{k,j} = 1 \text{ for any } k$$

where $\wp_i$ is the projection onto $\mathbb{C}\Omega_j$. The corresponding C^*-algebras are Toeplitz-Cuntz-Krieger algebras .

Matricially free Gaussians

Arrays of matricially free Gaussians operators

$$\omega_{i,j}(s) = \sqrt{d_j}(\wp_{i,j}(s) + \wp^*_{i,j}(s))$$

play the role of matricial semicircular operators , they satisfy

$$\omega(s) = \sum_{i,j} \omega_{i,j}(s)$$

and their arrays (rescaled if needed) generalize semicircular operators.

Hermitian Gaussian Symmetric Random Block Ensemble

Theorem 1

If $Y(u,n)$ are independent HGRM, then

$$T_{i,j}(s,n) \;\; \rightarrow \;\; \widehat{\omega}_{i,j}(s) = \widehat{\wp}_{i,j}(s) + \widehat{\wp}_{i,j}(s)^*$$

in the sense of moments under partial traces, where

$$\widehat{\wp}_{i,j}(s) = \begin{cases} \sqrt{d_j}\wp_{j,j}(s) & \text{if } i = j \\ \sqrt{d_i}\wp_{i,j}(s) + \sqrt{d_j}\wp_{j,i}(s) & \text{if } i \neq j \end{cases}$$

where $d_j = \lim_{n\to\infty} n_j/n$ are called asymptotic dimensions.

Typical computations

Typical computations under partial traces for $u \neq s$:

$$
\begin{aligned}
& \tau_k(n)(T_{j,k}(u,n)T_{i,j}(s,n)T_{i,j}(s,n)T_{j,k}(u,n)) \\
\to\ & \Psi_k(\widehat{\omega}_{j,k}(u)\widehat{\omega}_{i,j}(s)\widehat{\omega}_{i,j}(s)\widehat{\omega}_{j,k}(u)) \\
=\ & \Psi_k(\wp_{j,k}(u)^*\wp_{i,j}(s)^*\wp_{i,j}(s)\wp_{j,k}(u)) \\
=\ & d_i d_j \\
& \tau_k(n)(T_{i,k}(s,n)T_{i,k}(s,n)T_{j,k}(u,n)T_{j,k}(u,n)) \\
\to\ & \Psi_k(\widehat{\omega}_{i,k}(s)\widehat{\omega}_{i,k}(s)\widehat{\omega}_{j,k}(u)\widehat{\omega}_{j,k}(u)) \\
=\ & \Psi_k(\wp_{i,k}(s)^*\wp_{i,k}(s)\wp_{j,k}^*(u)\wp_{j,k}(u)) \\
=\ & d_i d_j
\end{aligned}
$$

Types of blocks

The symmetric block $T_{i,j}(s, n)$ is called

1. balanced if $d_i > 0$ and $d_j > 0$,
2. unbalanced if $d_i = 0 \wedge d_j > 0$ or $d_i > 0 \wedge d_j = 0$,
3. evanescent if $d_i = 0$ and $d_j = 0$.

Special cases

Special cases

In the general formula for mixed moments, we get

1. $T_{i,j}(s,n) \to \widehat{\omega}_{i,j}(s)$ if block is balanced
2. $T_{i,j}(s,n) \to \omega_{i,j}(s)$ if block is unbalanced, $j = 0 \wedge i > 0$
3. $T_{i,j}(s,n) \to \omega_{j,i}(s)$ if block is unbalanced, $j > 0 \wedge i = 0$
4. $T_{i,j}(s,n) \to 0$ if block is evanescent

Block Ginibre Ensemble

Theorem 2

If $Y(s,n)$ are complex independent GRM, then

$$\lim_{n\to\infty} T_{i,j}(s,n) = \eta_{i,j}(s)$$

in the sense of *-moments under partial traces, where

$$\eta_{i,j}(s) = \widehat{\wp}_{i,j}(2s-1) + \widehat{\wp}^{*}_{i,j}(2s)$$

are called matricial circular operators .

Canonical noncommutative random variables

In order to get the asymptotics of a more general class of HRM, we need more general random variables.

Definition

Let $\mu(s)$ be a probability measure on the real line whose free cumulants are $(r_k(s))_{k\geqslant 1}$, respectively. The formal sums

$$\gamma(s) = \ell_s^* + \sum_{k=0}^{\infty} r_{k+1}(s)\ell_s^k$$

are called canonical noncommutative random variables. If $\sum_{k=0}^{\infty} |r_{k+1}(s)| < \infty$, then $\gamma(s)$ is a bounded operator on the free Fock space.

Canonical matricial noncommutative random variables

Proposition

If $\sum_{k=0}^{\infty}|r_{k+1}(s)|<\infty$, then the canonical noncommuative random variable $\gamma(s)$ has the decomposition

$$\gamma(s)=\sum_{p,q=1}^{r}\gamma_{p,q}(s),$$

for any $r\in\mathbb{N}$, where

$$\begin{aligned}\gamma_{p,q}(s) &= \wp_{p,q}(s)^{*}+\delta_{p,q}r_{1}(s)P_{q}\\ &+ \sum_{k=1}^{\infty}r_{k+1}(s)\sum_{q_1,\ldots,q_{k-1}}\wp_{p,q_1}(s)\wp_{q_1,q_2}(s)\ldots\wp_{q_{k-1},q}(s)\end{aligned}$$

where $I=P_1+\ldots+P_r$ is the decomposition of the identity on $\mathcal{M}$. Symmetrized operators are denoted by $\widehat{\gamma}_{p,q}(s)$.

Generalization for Hermitian Symmetric Block Ensemble

Theorem 3

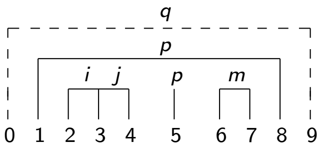
Let $\{Y(s,n) : s \in S, n \in \mathbb{N}\}$ be a family of independent Hermitian random matrices whose asymptotic joint distribution under $\tau(n)$ agrees with that of the family $\{\gamma(s) : s \in S\}$ under $\Psi = \sum_q d_q \Psi_q$ and which is asymptotically free from $\{D_1, \ldots, D_r\}$. If $\| Y(s,n) \| \leqslant C$ almost everywhere for any n, s and some C, then

$$T_{p,q}(s,n) \to \widehat{\gamma}_{p,q}(s)$$

as $n \to \infty$ in the sense of moments under partial traces.

Limit mixed moments can be expressed in terms of colored noncrossing partitions. Here is an example with blocks:

$$\pi_1 = \{1, 8\}, \ \ \pi_2 = \{2, 3, 4\}, \ \ \pi_3 = \{5\}, \ \ \pi_4 = \{6, 7\}$$



Contributions from blocks:

$$b(\pi_1, f) = d_p r_2, \ \ b(\pi_2, f) = d_i d_j r_3, \ \ b(\pi_3, f) = r_1, \ \ b(\pi_4, f) = d_m r_2$$

where $r_j = r_j(u)$ are free cumulants.

Asymptotic polynomials

Corollary

Asymptotic mixed moments are polynomials in asymptotic dimensions of the form

$$\Psi_q\left(\widehat{\gamma}_{p_1,q_1}(s_1)\ldots\widehat{\gamma}_{p_m,q_m}(s_m)\right)$$

$$= \sum_{\pi\in\mathcal{NC}_m((w_1,s_1),\ldots,(w_m,s_m))} \prod_{\text{blocks } \pi_k} r_{|\pi_k|}(s^{(k)}) \prod_{j\in w_1\cup\ldots\cup w_m} d_j^{|B_j(\pi)|}$$

where the summation runs over the set of suitably defined noncrossing colored partitions (matricially) adapted to all indices $w_j = \{p_j, q_j\}$ and s_j and $B_j(\pi)$ is the set of subblocks of blocks of π of the form $\{i, i+1\}$ which are colored by j and $s^{(k)} = s_i$ for all $i \in \pi_k$.

Applications

Let us present three applications:

1. (A1) products of independent GRM and Fuss-Narayana polynomials
2. (A2) random matrix model for monotone independence
3. (A3) random matrix model for free Meixner laws and conditionsl independence

A1: Products of independent GRM

Asymptotic moments of products of independent GRM

Let

$$B(n) = T_{1,2}(n)T_{2,3}(n)\ldots T_{p,p+1}(n)$$

for any $n \in \mathbb{N}$. Then, for any $k \in \mathbb{N}$,

$$\lim_{n\to\infty} \tau_1(n)\left((B(n)B^*(n))^k\right) = P_k(d_1, d_2, \ldots, d_{p+1})$$

where $d_1, d_2, \ldots, d_{p+1}$ are asymptotic dimensions and P_k's are some multivariate polynomials.

A1: Multivariate Fuss-Narayana polynomials

Multivariate Fuss-Narayana polynomials

The polynomials P_k have the explicit form

$$P_k(d_1, \ldots, d_{p+1}) = \sum_{j_1+\ldots+j_{p+1}=pk+1} N(k, j_1, \ldots, j_{p+1})\, d_1^{j_1} d_2^{j_2} \ldots d_{p+1}^{j_{p+1}}$$

and are called multivariate Fuss-Narayana polynomials and their coefficients are given by

$$N(k, j_1, \ldots, j_{p+1}) = \frac{1}{k}\binom{k}{j_1+1}\binom{k}{j_2}\ldots\binom{k}{j_p}.$$

If $p = 1$, we get so-called Narayana polynomials .

A1: Marchenko-Pastur law

Marchenko-Pastur law

The Marchenko-Pastur law with shape parameter $t > 0$ is given by

$$\pi_t = \max\{1 - t, 0\}\delta_0 + \frac{\sqrt{(x-a)(b-x)}}{2\pi x} \mathbb{1}_{[a,b]}(x)dx$$

where $a = (1 - \sqrt{t})^2$ and $b = (1 + \sqrt{t})^2$.

Corollary 1

The moments of the free convolution of Marchenko-Pastur laws with shape parameters $t_1, \ldots, t_p$,

$$\pi_{t_1} \boxtimes \pi_{t_2} \boxtimes \ldots \boxtimes \pi_{t_p}$$

are given by

$$P_k(1, t_1, \ldots, t_p)$$

where $k \in \mathbb{N}$.

A1: Fuss-Catalan numbers

Corollary 2

If $d_1 = \ldots = d_{p+1} = d$, then we get

$$P_k(d_1, \ldots, d_{p+1}) = d^{kp} F(p,k)$$

where

$$F(p,k) = \frac{1}{mp+1}\binom{mp+m}{m}$$

are Fuss-Catalan numbers (Alexeev, Götze and Tikhomirov).

A1: Free Bessel laws

Corollary 3

If $d_1 = \ldots = d_p = 1$ and $d_{p+1} = t$, then $P_k(d_1, \ldots, d_{p+1})$ are the moments of the free Bessel laws

$$\pi^{\boxtimes(p-1)} \boxtimes \pi^{\boxplus t}$$

where $\pi = \pi_1$ is the standard MP distribution.

A2: Random matrix model for monotone independence

Consider HRM of the block form

$$Y(s,n) = \left(\begin{array}{cc} A(s,n) & B(s,n) \\ C(s,n) & D(s,n) \end{array} \right)$$

where $s \in \{1,2\}$ and

1. the sequences $(D(s,n))$ are *balanced*,
2. the sequences of symmetric blocks built from $(B(s,n))$ and $(C(s,n))$ are *unbalanced*,
3. the sequences $(A(s,n))$ are *evanescent*,

Asymptotic monotone independence

Identifying blocks with their canonical embeddings in $M_n(\mathbb{C})$, the pair $\{B(1,n) + C(1,n), Y(2,n)\}$ is asymptotically monotone independent with respect to $\tau_1(n)$.

A3: Fock space for free Meixner laws

Consider now the special case of the matricially free Fock space

$$\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2,$$

where

$$\begin{aligned} \mathcal{M}_1 &= \mathbb{C}\Omega_1 \oplus \bigoplus_{k=0}^{\infty} (\mathcal{H}_2^{\otimes k} \otimes \mathcal{H}_1), \\ \mathcal{M}_2 &= \mathbb{C}\Omega_2 \oplus \bigoplus_{k=1}^{\infty} \mathcal{H}_2^{\otimes k}, \end{aligned}$$

and Ω_1, Ω_2 are unit vectors, $\mathcal{H}_j = \mathbb{C}e_j$ for $j \in \{1, 2\}$, where e_1, e_2 are unit vectors.

A3: Gaussian operators for free Meixner laws

Use simplified notation

$$\wp_1 = \sqrt{\beta_1}\wp_{2,1}, \quad \wp_2 = \sqrt{\beta_2}\wp_{2,2}$$

for the rescaled matricially free creation operators. Let

$$\omega_1 = \omega_{2,1}, \quad \omega_2 = \omega_{2,2}$$

be the associated matricially free Gaussian operators.

A3: Moments of free Meixner laws

Moments of free Meixner laws

If μ is the free Meixner law corresponding to $(\alpha_1, \alpha_2, \beta_1, \beta_2)$, where $\beta_1 \neq 0$ and $\beta_2 \neq 0$, then its m-th moment is given by

$$M_m(\mu) = \Psi_1((\omega + \gamma)^m),$$

where

$$\omega = \omega_1 + \omega_2$$

and

$$\gamma = (\alpha_2 - \alpha_1)(\beta_1^{-1}\wp_1\wp_1^* + \beta_2^{-1}\wp_2\wp_2^*) + \alpha_1,$$

and Ψ_1 is the state defined by the vector Ω_1.

A3: Random matrix model for free Meixner laws

Random matrix model for free Meixner laws

Let $\beta_1 = v_{2,1} > 0$ and $\beta_2 = v_{2,2} > 0$ be the variances in blocks C and D. Then

$$\lim_{n\to\infty} \tau_1(n)\left((M(s,n))^m\right) = \Psi_1((\omega+\gamma)^m)$$

where

$$M(s,n) = Y(s,n) + \alpha_1 D_1 + \alpha_2 D_2$$

for any $n \in \mathbb{N}$, where $I(n) = D_1 + D_2$ is the decomposition of the $n \times n$ unit matrix.

A3: Asymptotic conditional freeness

Asymptotic conditional freeness

The Free Meixner Ensemble

$$\{M(s,n) : s \in S, n \in \mathbb{N}\}$$

is asymptotically conditionally free with respect to the pair of partial traces $(\tau_1(n), \tau_2(n))$.

Papers

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