SMOOTHNESS OF CARRYING SIMPLICES FOR THREE-DIMENSIONAL COMPETITIVE SYSTEMS: A COUNTEREXAMPLE

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Abstract: For a dissipative totally competitive system of ODEs $\dot{x}^i = x^i f^i(x)$, $\partial f^i/\partial x^j < 0$, $i, j = 1, 2, 3$, on the nonnegative octant K in $\mathbb{R}^3$ for which 0 is a repeller, M. W. Hirsch proved the existence of an invariant unordered Lipschitz surface (the carrying simplex) attracting all points in $K \setminus \{0\}$. We give an example (of a Lotka–Volterra type) showing that the carrying simplex need not be of class C^1.

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1 Introduction

A three-dimensional system of C^1 ordinary differential equations (ODEs)

$$\dot{x}^i = x^i f^i(x), \tag{S}$$

where $f = (f^1, f^2, f^3)\colon K \to \mathbb{R}^3$, $K := \{x = (x^1, x^2, x^3) \in \mathbb{R}^3 : x^i \geqslant 0 \text{ for } i = 1, 2, 3\}$ is called *totally competitive* if

$$\frac{\partial f^i}{\partial x^j}(x) < 0$$

for all $x \in K$, $i, j = 1, 2, 3$. We write $F = (F^1, F^2, F^3)$ with $F^i(x) = x^i f^i(x)$. The symbol $DF(x)$ denotes the Jacobian matrix of the vector field F at $x \in K$, $DF(x) = [(\partial F^i/\partial x^j)(x)]_{i,j=1}^3$.

Denote by $\phi = \{\phi_t\}$ the local flow generated on K by (S): $\phi_t(x_0)$ is the value at time t of the unique solution of the initial value problem $\dot{x}^i = F^i(x)$, $x(0) = x_0$. We identify the tangent bundle of K with the product bundle $K \times V$, where $V := \{v = (v^1, v^2, v^3)\}$ stands for the vector space of all three-vectors, with the Euclidean norm $\|\cdot\|$. We write $D\phi = \{D\phi_t\}$ for the family of local vector bundle mappings generated by the linearization of system (S) on $K \times V$: for any $x_0 \in K$, $v_0 \in V$ and $t \in \mathbb{R}$ such that $\phi_t(x_0)$ is defined, the symbol $D\phi_t(x_0)v_0$ denotes the value at time t of the unique solution of the initial value problem $\dot{v} = DF(\phi_t(x_0))v$, $v(0) = v_0$. Let $V_+ := \{v \in V : v^i \geqslant 0 \text{ for } i = 1, 2, 3\}$, $V_{++} := \{v \in V : v^i > 0 \text{ for } i = 1, 2, 3\}$. Define K° to be the set of points in K with positive coordinates, that is, the interior of K in $\mathbb{R}^3$. Set ∂K to be the boundary of K in $\mathbb{R}^3$, $\partial K = K \setminus K^\circ$. We say that system (S) is *dissipative* if there is a compact invariant set $\Gamma \subset K$ attracting all bounded subsets of K. For a dissipative system, both ϕ and $D\phi$ are defined for all $t \geqslant 0$. A compact invariant set $A \subset K$ is *repelling* if $\alpha(B) \subset A$ for some neighborhood B of A in K. (For the definitions of concepts from the theory of dynamical systems see Hale [3].)

The theory of totally competitive systems (in arbitrary dimension) was initiated by M. W. Hirsch in [7] (see also his previous papers [5], [6]). He obtained, among others, the following result (we formulate it for dimension 3).

Theorem 1.1 *Assume that* (S) *is a dissipative three-dimensional totally competitive system of ODEs such that* 0 *is a repelling equilibrium. Then there exists a compact invariant set* Σ *with the following properties:*

(a) Σ *is homeomorphic via radial projection to the standard two-dimensional probability simplex* $\{x \in K : \sum_{i=1}^{3} x^i = 1\}$.

(b) *Let* P_v *denote the orthogonal projection along* $v \in V_{++}$. *Then* $P_v|_\Sigma$ *is a Lipeomorphism onto its image.*

(c) *For each* $x \in K \setminus \{0\}$, $\omega(x) \subset \Sigma$.

Following M. L. Zeeman [14] we refer to Σ as the *carrying simplex* for (S). We write Σ° for $\Sigma \cap K^\circ$, and $\partial\Sigma$ for $\Sigma \cap \partial K$. Obviously $\partial\Sigma = \Sigma \setminus \Sigma^\circ$.

Let $P := P_{(1,1,1)}$. By Theorem 1.1(b) the mapping $R := (P|_\Sigma)^{-1}$ is well defined and Lipschitz.

For $x \in \Sigma$ denote by $\mathcal{C}_x$ the *tangent cone* of Σ at x, $\mathcal{C}_x := \{\alpha v : \alpha \geqslant 0,$ there is a sequence $\{x_n\} \subset \Sigma \setminus \{x\}, x_n \to x$ as $n \to \infty$, such that $(x_n - x)/\|x_n - x\| \to v\}$. As Σ is invariant, for all $x \in \Sigma$ and $t \in \mathbb{R}$ one has $D\phi_t(x)\mathcal{C}_x = \mathcal{C}_{\phi_t(x)}$. From Theorem 1.1(b) we derive that $\mathcal{C}_x \cap V_{++} = \emptyset$ for each $x \in \Sigma$.

An important subclass of dissipative totally competitive systems for which 0 is a repelling equilibrium is the Lotka–Volterra systems of the form

$$\dot{x}^i = b_i x^i \left(1 - \sum_{j=1}^{3} a_{ij} x^j\right)$$

where $a_{ij} > 0$ and $b_i > 0$. For an introduction to Lotka–Volterra systems and their application in biology see the book [9] by J. Hofbauer and K. Sigmund.

M. W. Hirsch asked in [7] about conditions for Σ to be of class C^1. P. Brunovský was the first to tackle the problem: he showed in [2] that under some hyperbolicity assumptions Σ° is a C^1 manifold. From recent results of I. Tereščák [13] it follows that if $A \subset \Sigma$ is a repeller in Σ then the restriction $P|_{B(A)}$, where $B(A)$ stands for the repulsion basin of A, is a C^1 diffeomorphism. M. Benaïm in [1] has given criteria for $P|_{B(A)}$ to be of class $C^{k+\alpha}$, $k = 1, \ldots$, $\alpha \in [0, 1)$. In all those results the centerpiece of the proof is the observation that the negativity of the Jacobian matrix implies that $D\phi_t(x)$ sends $V_+ \setminus \{0\}$ to V_{++}, for all $t < 0$ and all $x \in B(A)$. That in turn gives the existence of a decomposition of the tangent bundle of $\mathbb{R}^3$ restricted to A, that is, $A \times \mathbb{R}^3$, into the direct sum of two invariant subbundles, a one-dimensional $\mathcal{U}$ and a complementary $\mathcal{V}$. Furthermore, the subbundle $\mathcal{V}$ attracts all one-codimensional tangent subspaces close to it, which allows one to prove, using the theory of invariant manifolds, that $B(A)$ is a smooth manifold, with $\mathcal{V}$ serving as the tangent bundle of A.

In general, the above method breaks down when we pass to the boundary $\partial\Sigma$ of Σ: For $x \in \partial\Sigma$ and $t < 0$ one can prove only that $D\phi_t(x)$ takes V_+ into itself. In his papers [10] and [11] the author gave (ecologically relevant) conditions for Σ to be a C^1 manifold-with-corners, neatly embedded in K. (Roughly speaking, they mean that for each subsimplex, no part of its boundary is attracting.) Now, one can extend the invariant bundle decomposition (as in the previous paragraph) over the whole of Σ and apply the theory of invariant manifolds. (We note in passing that all the results of Brunovský, Tereščák, Benaïm and the present author mentioned above are valid for arbitrary dimensions.)

In the present note we give an example of a Lotka–Volterra totally competitive three-dimensional system of ODEs satisfying all the requirements of Theorem 1.1 for which the carrying simplex Σ is not continuously differentiable at (a part of) its boundary.

Denote by E the set of equilibria for (S). If $x \in E \setminus \{0\}$ then from the Perron–Frobenius theory (see, e.g., Seneta [12]) applied to $\exp(-DF(x))$ we deduce that an eigenvalue μ of the Jacobian matrix $DF(x)$ with the smallest real part is real, and there is an eigenvector pertaining to μ contained in V_+.

An equilibrium x is called *axial* if only one of its coordinates is positive, and *planar* if precisely two of its coordinates are positive.

For $i = 1, 2, 3$ we write $V^i := \{v \in V : v^i = 0\}$. For $i, j = 1, 2, 3$, $i \neq j$, we write $V^{ij} := \{v \in V : v^i = v^j = 0\}$. Similarly, $K^i := \{x \in K : x^i = 0\}$, $K^{ij} := \{x \in K : x^i = x^j = 0\}$. Evidently, $\Sigma^i := \Sigma \cap K^i$, $\Sigma^{ij} := \Sigma \cap K^{ij}$.

2 An example

Consider the three-dimensional Lotka–Volterra system

$$\begin{aligned} \dot{x}^1 &= x^1(1 - x^1 - a_{12}x^2 - a_{13}x^3) \\ \dot{x}^2 &= x^2(1 - a_{21}x^1 - x^2 - a_{23}x^3) \\ \dot{x}^3 &= x^3(1 - a_{31}x^1 - a_{32}x^2 - x^3) \end{aligned} \tag{2.1}$$

with

$$\begin{aligned} 0 < a_{21} < 1 < a_{12} \\ 0 < a_{23} < 1 < a_{32} \\ 0 < a_{13} < 1 < a_{31} \end{aligned} \tag{2.2}$$

and

$$a_{12} < 2,\ a_{32} < 2,\ a_{31} > 2,\ a_{12} \neq a_{32}. \tag{2.3}$$

Condition (2.2) yields that system (2.1) has no planar equilibria. Obviously, the axial equilibria are $y_1 = (1, 0, 0)$, $y_2 = (0, 1, 0)$, $y_3 = (0, 0, 1)$.

Denote by Σ the carrying simplex for (2.1). The invariant set Σ^i is the carrying simplex for system (2.1) restricted to K^i. By the analog of Theorem 1.1 for dimension two (see [7]), Σ^i is a Lipschitz manifold-with-boundary homeomorphic to the real interval $[-1, 1]$.

Now we look closer at the sets Σ^3 and Σ^2 in the vicinity of y_1. The matrix $DF(y_1)$ has the form

$$\begin{bmatrix} -1 & -a_{12} & -a_{13} \\ 0 & 1 - a_{21} & 0 \\ 0 & 0 & 1 - a_{31} \end{bmatrix}.$$

The smaller eigenvalue of the restriction of $DF(y_1)$ to the invariant coordinate subspace V^3 equals -1 and corresponds to the eigenvector $(1, 0, 0)$. The larger eigenvalue equals $1 - a_{21} > 0$ and corresponds to the eigenvector

$$w_{12} := \frac{(1, (a_{21} - 2)/a_{12}, 0)}{\|(1, (a_{21} - 2)/a_{12}, 0)\|}.$$

Consequently, for the restriction of system (S) to K^3 the Jacobian matrix of the (restricted) vector field at y_1 has one positive and one negative eigenvalue. By the theory of invariant manifolds (see, e.g., Hirsch, Pugh and Shub [8]) there are a unique locally invariant one-dimensional C^1 stable manifold $M^{\mathrm{s}}(y_1)$ tangent at y_1 to $(1,0,0)$ and a unique locally invariant one-dimensional C^1 unstable manifold $M^{\mathrm{u}}(y_1)$ tangent at y_1 to w_{12}. As the half-line K^{23} is invariant and tangent at y_1 to $(1,0,0)$, one has $M^{\mathrm{s}}(y_1) \subset K^{23}$. By the Grobman–Hartman theorem (see, e.g., Thm. IX.7.1 in Hartman [4]) the phase portrait of the restricted dynamical system $\{\phi_t|_{K^3}\}$ close to y_1 is the same as that of its linearization $\dot{\xi} = (DF(y_1)|_{V^3})\xi$. Therefore, any locally invariant topological manifold-with-boundary passing through y_1 and contained in Σ^3 is a subset of $M^{\mathrm{s}}(y_1) \cup M^{\mathrm{u}}(y_1)$. The intersection $\Sigma^3 \cap K^{23}$ equals $\{y_1\}$, hence there is a neighborhood U of y_1 such that $\Sigma^3 \cap U = M^{\mathrm{u}}(y_1) \cap U$, in particular Σ^3 is tangent at y_1 to w_{12}.

The smaller eigenvalue of the restriction of the matrix $DF(y_1)$ to the invariant subspace V^2 equals $1 - a_{31} < -1$ and corresponds to the eigenvector

$$w_{13} := \frac{(1,0,(a_{21}-2)/a_{12})}{\|(1,0,(a_{21}-2)/a_{12}\|}.$$

The larger eigenvalue, -1, corresponds, of course, to $(1,0,0)$. Thus for the restriction of system (S) to K^2 the Jacobian matrix of the (restricted) vector field at y_1 has two simple negative eigenvalues. The theory of invariant manifolds guarantees the existence of a neighborhood U of y_1 such that for each $x \in K^2 \cap U$, $x \neq y_1$, the vectors $(y_1 - \phi_t(x))/\|y_1 - \phi_t(x)\|$ converge, as $t \to \infty$, either to $\pm(1,0,0)$ or to $\pm w_{13}$. Moreover, the latter is the case if and only if x belongs to a unique locally invariant one-dimensional strongly stable C^1 manifold $M^{\mathrm{ss}}(y_1)$ tangent at y_1 to w_{13}. But this is impossible for $x \in \Sigma^2$, as then for t sufficiently large one would have $(\phi_t(x))^1 > 1 = y_1^1$ and $(\phi_t(x))^2 > 0 = y_1^2$, which contradicts the analog of Theorem 1.1 for dimension two. Consequently, Σ^2 is tangent at y_1 to the half-line K^{23}.

We investigate now the vicinity of y_2. The matrix of $DF(y_2)$ has the form

$$\begin{bmatrix} 1-a_{12} & 0 & 0 \\ -a_{21} & -1 & -a_{23} \\ 0 & 0 & 1-a_{32} \end{bmatrix}.$$

The smaller eigenvalue of the restriction of $DF(y_2)$ to the invariant coordinate subspace V^3 equals -1 and corresponds to the eigenvector $(0,1,0)$. The larger eigenvalue, $1 - a_{12}$, is negative and corresponds to the eigenvector

$$w_{21} := \frac{((a_{12}-2)/a_{21},1,0)}{\|((a_{12}-2)/a_{21},1,0)\|}.$$

Arguing as in the above paragraph we prove that there is a neighborhood U of y_2 such that for each $x \in K^3 \cap U$, $x \neq y_2$, the vectors $(y_2 - \phi_t(x))/\|y_2 - \phi_t(x)\|$ converge, as $t \to \infty$, either to $\pm(0,1,0)$ or to $\pm w_{21}$, and the former is the case if and only if x belongs to a unique locally invariant one-dimensional strongly stable C^1 manifold $M^{\mathrm{ss}}(y_2)$ tangent at y_2 to $(0,1,0)$. By uniqueness, $M^{\mathrm{ss}}(y_2) \cap U = K^{13} \cap U$. As the intersection $\Sigma^3 \cap K^{13}$ equals $\{y_2\}$, we have that Σ^3 is tangent at y_2 to w_{21}.

Finally, the smaller eigenvalue of the restriction of the matrix $DF(y_2)$ to the invariant coordinate subspace V^1 equals -1 and corresponds to the eigenvector $(0,1,0)$. The larger eigenvalue, $1 - a_{32}$, is negative and corresponds to the eigenvector

$$w_{23} := \frac{(0,1,(a_{32}-2)/a_{23})}{\|(0,1,(a_{32}-2)/a_{23})\|}.$$

Precisely as above we prove that Σ^1 is tangent at y_2 to w_{23}. (It should be remarked here that, since system (2.1) is analytic (hence C^2) and its restrictions to Σ^i are two-dimensional, one could use results on differentiable conjugacy instead of the Grobman–Hartman theorem. However, such an argument may not be valid for a C^1 perturbation of the system; compare the remarks at the end of the paper.)

Suppose for contradiction that the mapping R is of class C^1. This means that there is a relatively open neighborhood U of $P(\Sigma)$ in $(\mathrm{span}(1,1,1))^{\perp}$ and a C^1 mapping $\tilde{R}\colon U \to \mathbb{R}^3$ such that $\tilde{R}|_{P(\Sigma)} = R$. The two-dimensional vector subspaces $\mathrm{Im}(D\tilde{R}(P(x)))$ may be, at some $x \in \partial\Sigma$, different for different extensions $\tilde{R}$. However, if at some $x \in \partial\Sigma$ one has $\dim \mathrm{span}\, \mathcal{C}_x = 2$ then $\mathrm{Im}(D\tilde{R}(P(x))) = \mathrm{span}\, \mathcal{C}_x$ for any extension $\tilde{R}$ of R. It has been proved above that this holds at both the equilibria, y_1 and y_2, on Σ^3. If $x \in \Sigma^3$ is not an equilibrium then locally $P(\Sigma)$ is a Lipschitz two-dimensional manifold-with-boundary, where the (manifold) boundary is C^1 diffeomorphic to an open real interval. Consequently, $\mathcal{C}_x$ contains besides $\mathrm{span}\, F(x)$ some other vector. Therefore $\dim \mathrm{span}\, \mathcal{C}_x = 2$. We can legitimately write $\mathcal{T}_x := \mathrm{Im}(D\tilde{R}(P(x)))$ for each $x \in \Sigma^3$. The mapping $\Sigma^3 \ni x \mapsto \mathcal{T}_x$ is a continuous family of two-dimensional vector subspaces of V, that is, a two-dimensional subbundle of the product bundle $\Sigma^3 \times V$. By the invariance of the family of tangent cones the subbundle obtained is invariant, too: $\mathcal{T}_{\phi_t(x)} = D\phi_t(x)\mathcal{T}_x$ for all $x \in \Sigma$ and $t \in \mathbb{R}$.

Our proof will rest on the observation that there cannot exist an invariant two-dimensional subbundle $\mathcal{T}$ of $\Sigma^3 \times V$ with $\mathcal{T}_{y_1} = \mathrm{span}\{(1,0,0), w_{12}\} = V^3$ and $\mathcal{T}_{y_2} = \mathrm{span}\{w_{21}, w_{23}\}$. Notice that the subbundle $\Sigma^3 \times V^3$ is invariant. At y_2 the two-dimensional vector subspaces V^3 and $\mathcal{T}_{y_2}$ are transverse, as $V^3 \cap \mathrm{span}\{w_{21}, w_{23}\} = \mathrm{span}\, w_{21}$. By continuity, $\mathcal{T}_x$ is transverse to V^3 at points $x \in \Sigma^3$ sufficiently close to y_2. Invariance yields that $\mathcal{T}_x$ is transverse to V^3 at all $x \in \Sigma^3$, $x \neq y_1$.

Since $D\phi_t(y_1) = \exp(t\, DF(y_1))$ for all $t \in \mathbb{R}$, it follows from the spectral properties of $DF(y_1)$ that for each $t > 0$ the spectral radius of $D\phi_{-t}(y_1)|_{V^3}$ is smaller than $\|D\phi_{-t}(y_1)w_{13}\|$. Take $T > 0$ so large that

$$\frac{\|D\phi_{-T}(y_1)w_{13}\|}{\|D\phi_{-T}(y_1)|_{V^3}\|} > 2.$$

Let U be a neighborhood of y_1 such that $\phi_{-t}(\Sigma^3 \cap U) \subset \Sigma^3 \cap U$ for all $t \geqslant 0$ (recall that y_1 is a repelling point in Σ^3). Take a continuous mapping $\Sigma^3 \cap U \ni x \mapsto w(x) \in V \setminus V^3$ such that $w(y_1) = w_{13}$. For each $x \in \Sigma^3 \cap U$ we can write $V = \mathcal{S}_x \oplus V^3$, where $\mathcal{S}_x := \operatorname{span} w(x)$. The linear isomorphism $D\phi_{-T}(x)$ can be written in the above decomposition in the matrix form

$$\begin{bmatrix} A(x) & 0 \\ B(x) & C(x) \end{bmatrix}.$$

By taking U smaller, if necessary, we can assume $\|A(x)\|/\|C(x)\| > 2$ for all $x \in \Sigma^3 \cap U$.

Let $\Pi_1(x)$ (resp. $\Pi_2(x)$) be the projection of V onto $\mathcal{S}_x$ along V^3 (resp. onto V^3 along $\mathcal{S}_x$). For any $x \in \Sigma^3 \cap U$ and $v \in V$ we estimate

$$\begin{aligned} \frac{\|\Pi_1(\phi_{-T}(x))D\phi_{-T}(x)v\|}{\|\Pi_2(\phi_{-T}(x))D\phi_{-T}(x)v\|} &\geqslant \frac{\|A(x)\|\,\|\Pi_1(x)v\|}{\|C(x)\|\,\|\Pi_2(x)v\| + \|B(x)\|\,\|\Pi_1(x)v\|} \\ &= \frac{\frac{\|A(x)\|}{\|C(x)\|}\frac{\|\Pi_1(x)v\|}{\|\Pi_2(x)v\|}}{1 + \frac{\|B(x)\|}{\|C(x)\|}\frac{\|\Pi_1(x)v\|}{\|\Pi_2(x)v\|}}. \end{aligned} \tag{2.4}$$

We have already proved that for some $x \in \Sigma^3 \cap U$, $x \neq y_1$, there is a vector $v \in \mathcal{T}_x \setminus V^3$. Denote, for $n = 0, 1, \ldots$,

$$a_n := \frac{\|\Pi_1(\phi_{-nT}(x))D\phi_{-nT}(x)v\|}{\|\Pi_2(\phi_{-nT}(x))D\phi_{-nT}(x)v\|}.$$

As $v \notin V^3$ and V^3 is invariant, we have $a_n > 0$ for all n (the case $a_n = \infty$ is not excluded). Since $\phi_{-nT}(x)$ tends to y_1 as $n \to \infty$, for each $\varepsilon > 0$ we can find $N \in \mathbb{N}$ such that $\|B(\phi_{-nT}(x))\|/\|C(\phi_{-nT}(x))\| < \varepsilon$ for all $n \geqslant N$ (recall that $B(y_1) = 0$ and $C(y_1) > 0$). From (2.4) it follows that

$$a_{n+1} > \frac{2a_n}{1 + \varepsilon a_n}$$

for $n \geqslant N$. We claim that there is a subsequence a_{n_k} tending to ∞ as $k \to \infty$. Indeed, if not then the set $\{a_n\}$ would be bounded, say $a_n < A$ for all $n \in \mathbb{N}$.

Taking $\varepsilon < 1/3A$ we get

$$\frac{a_{n+1}}{a_n} > \frac{2}{1 + \frac{a_n}{3A}} > \frac{3}{2}$$

for sufficiently large n, a contradiction.

We have that $D\phi_{-n_k T}(x)v/\|D\phi_{-n_k T}(x)v\|$ converges to $\pm w_{13}$ as $k \to \infty$. Consequently, w_{13} belongs to $\mathcal{T}_{y_1}$, which is impossible.

It should be remarked that in the proof above the only thing that matters (besides total competitivity etc.) is inequalities between eigenvalues of the linearization of the vector field F at the (unique) axial equilibria on K^{23} and K^{13} (plus nonexistence of planar equilibria on K^3). The choice of a Lotka–Volterra system is due only to the simplicity of the equations. Also, as these inequalities are strong, it follows that the property described in this note is robust (it persists under C^1 perturbations of the function f).

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