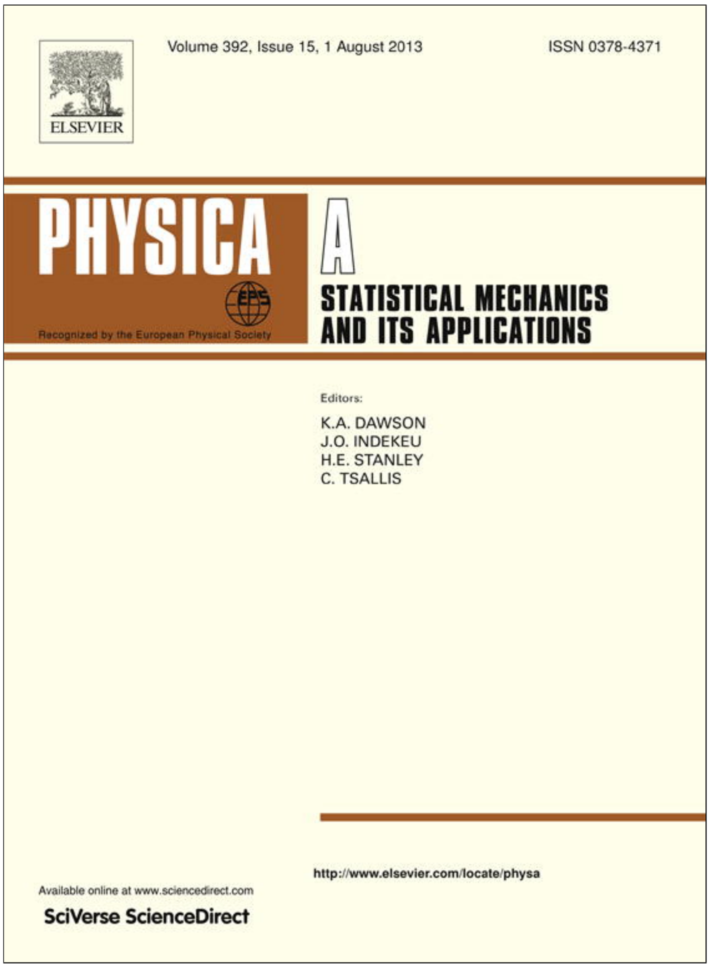
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Tempered stable Lévy motion driven by stable subordinator

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HIGHLIGHTS

- The subordinated tempered stable Lévy motion with inverse stable subordinator is examined.
- We analyze the main statistical properties of the considered system.
- We propose the parameter's estimation procedure.
- We analyze the ensemble averaged MSD as a tool of recognition between subdiffusive and diffusive models.
- We examine real financial data by using the considered processes.

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ABSTRACT

In this article we propose a new model for financial data description. Combining two independent mechanisms, namely the tempered stable process and inverse stable subordinator, we obtain a new model which captures not only the tempered stable character of the underlying data but also such a property as periods in which the values of an asset stay on the same level. Moreover, we classify our system to the family of subdiffusive processes and investigate its tail behavior. We describe in detail testing and estimation procedures for the proposed model. In the last step we calibrate our model to the real data.

1. Introduction

The idea of subordination was introduced by Bochner in 1949 in Ref. [1] and explored further by many authors, see for instance [2]. But since the invention of the technique of subordination through inverse subordinators, the number of papers from this field has grown rapidly. One should mention here especially the phenomenon of subdiffusion [3], where the particle motion is slowed down compared to the normal diffusion. Anomalous behavior was observed in a variety of physical systems. One should mention here charge carrier transport in amorphous semiconductors [4–6], intracellular transport [7], and the motion of molecules inside E. coli cells [8]. As it was mentioned above, one of the methods which leads to the description of the phenomenon of subdiffusive behavior is expressed in the language of inverse subordinators, namely we change the time of the underlying system $B(t)$ in a way that it is now modeled by some inverse subordinator $S(t)$, thus as a benchmark model we obtain $B(S(t))$. Usual choices of this new operational time are inverse stable [9,10] or inverse tempered stable [11,12] processes. Such a technique allows us to model trapping events, namely situations when the particle gets immobilized for some periods of time. Let us mention a wide area of applications of the subordinated models, for instance in geophysics [13] and statistical physics [14,15].

Behavior typical for subdiffusive processes one can also observe in finance. Especially in emerging markets, where the number of participants and as a consequence the number of transactions is rather low, one can observe periods when the

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prices of assets stay motionless [16]. This observation leads in consequence to development of various financial models with time change. We mention here arithmetic Brownian motion [17,18], geometric Brownian motion [19], and the α-stable Ornstein–Uhlenbeck process [20].

The general class of subordinated processes was examined in Ref. [2]. In this paper we consider the special case of such systems, namely we propose a subordinated tempered stable process as a model of financial data description. Thus we capture two very important properties of the present markets. One, the underlying process is not Gaussian as is reported for many financial time series, and, second, the process is time-changed thus we are able to model situations when the prices of an asset stay on the same level. Our new model fulfills those two very important properties, and thus serves as a new tool for data description. We should also mention here that the subdiffusive structure of the proposed system can be useful not only for financial data but also for a description of various phenomena.

The article is structured as follows. In Section 2 we review some basic facts of tempered stable distribution and indicate its relation with the α-stable one. In Section 3 we describe the model based on a subordinated tempered stable process. We derive its basic characteristics which allow us to classify it as a subdiffusive model. Moreover, we investigate tail properties of the proposed system and present the corresponding fractional Fokker–Planck equation. Section 4 is devoted to estimation and testing procedures for the considered process. In Section 5 we fit the examined model to real financial data. We estimate parameters and give an explanation why a tempered stable process is the most appropriate model in the considered case. Section 6 concludes the paper.

2. Tempered skew stable distribution

The class of tempered stable distributions is very rich and widely used in practice. It arises as an extension of the very well known class of stable processes. This extension is made through exponential tempering of the density of the stable process. The first successful attempt to form such a class of distributions was performed in Ref. [21], where the authors simply cut the probability density of the stable process over some threshold. Their ideas were further developed in Ref. [22], where exponential tempering was introduced. Scale invariant truncated Lévy processes were also proposed in Ref. [23] to characterize some physical systems with correlated stochastic variables. However the complete description of tempered stable distributions and related processes was given recently in Ref. [24] (see also Ref. [25]).

In this paper we consider one of the members of the tempered stable family. Let us define the skew tempered stable random variable T by its characteristic function in the following way:

$$\mathbb{E}e^{itT} = \begin{cases} \exp((\lambda - it)^{\alpha} - \lambda^{\alpha} + i\alpha\lambda^{\alpha-1}t) & \text{for } \alpha \neq 1 \\ \exp((\lambda - it)\log(\lambda - it) - \lambda\log(\lambda) + it(1 + \log(\lambda))) & \text{for } \alpha = 1. \end{cases} \tag{1}$$

The random variable T is known also as the truncated Lévy flight [26,27]. One of the reasons we use this specific kind of distribution (i.e. a skewed one) is the fact that we observe in real data behavior adequate for this distribution, see Section 5.

Let us mention that the α parameter is called the index of stability and takes values between 0 and 2 while the parameter $\lambda > 0$ is called the tempering index (or tempering parameter).

The most important property of a random variable T is its connection with the skew ($\beta = 1$) stable distribution, which is visible for instance in the relation between their Lévy measures. This relation leads also to the connection between probability density functions (PDFs), namely:

$$p_T(x) = \begin{cases} \exp(-\lambda x + (\alpha - 1)\lambda^{\alpha})p_U(x - \alpha\lambda^{\alpha-1}) & \text{for } \alpha \neq 1 \\ \exp(-\lambda x + \lambda)p_U(x - (1 + \log\lambda)) & \text{for } \alpha = 1, \end{cases} \tag{2}$$

where $p_T(\cdot)$ is the PDF of the tempered stable random variable T and $p_U(\cdot)$ is a PDF of random variable U with the stable distribution with index of stability α, skewness $\beta = 1$, mean $\mu = 0$ and scale parameter $\sigma = \left|\cos\frac{\pi\alpha}{2}\right|^{1/\alpha}$ when $\alpha \neq 1$ and $\sigma = \frac{\pi}{2}$ for $\alpha = 1$. It is worth mentioning that the tempered stable random variable T has all moments finite in contrast to the pure stable distribution, which makes processes based on the tempered stable distribution more plausible in many applications. The first two moments of the random variable T are as follows:

$$ET = 0, \qquad ET^2 = \alpha(\alpha - 1)\lambda^{\alpha-2}.$$

The other difference between tempered stable and pure stable distributions is the behavior of the right tails. The right tail of the tempered stable distribution with parameters α and λ behaves like $e^{-\lambda x}x^{-\alpha}$ in contrast to the stable distribution where the right tail can be approximated by the power function $x^{-\alpha}$.

3. Subordinated tempered stable Lévy motion

In this section we define the process called subordinated tempered stable Lévy motion. We investigate the basic properties of the aforementioned model, namely the mean square displacement (MSD), which will allow us to classify it as a subdiffusive model. We calculate also the asymptotic properties of this process. This leads us to give a precise formula for the tail behavior of the subordinated tempered stable process. Finally we derive the fractional Fokker–Planck equation with some general kernel which describes the PDF of the process. We also give a link between subordinated tempered stable Lévy motion and the properly rescaled continuous time random walk (CTRW) model.

The subordinated tempered stable process $\{X(t)\}_{t\geq 0}$ is defined as

$$X(t) = T(S(t)), \tag{3}$$

where $\{T(t)\}$ is the tempered stable Lévy process (i.e. process of independent stationary increments with tempered stable distribution described in the previous section) and $\{S(t)\}$ is the inverse stable subordinator which is defined as follows [9]:

$$S(t) = \inf\{\tau > 0 : U(\tau) > t\}, \quad t \geq 0, \tag{4}$$

where $\{U(\tau)\}$ is the γ-stable Lévy process defined via its Laplace transform:

$$E(\mathrm{e}^{-uU(\tau)}) = \mathrm{e}^{-\tau u^{\gamma}}, \tag{5}$$

where $0 < \gamma < 1$.

The process $\{X(t)\}_{t\geq 0}$ is called a subordinated or time changed tempered stable Lévy system. Processes with time change become extremely popular in the description of so-called anomalous diffusion, therefore in this paper we examine some basic properties of the process defined in (3). The basic statistics which classifies processes is the mean square displacement (MSD), i.e. the second moment of the underlying process. If the MSD behaves as t^{β} where $\beta < 1$ we have subdiffusion, if $\beta > 1$ then we can say that the examined process exhibits superdiffusive behavior, [3]. The case with $\beta = 1$ corresponds to normal diffusion, namely the classical Brownian motion [3]. We assume that the processes $\{T(t)\}$ and $\{S(t)\}$ are independent. Thus from the total probability formula, the PDF of the process $X(t)$ satisfies:

$$f(x,t) = \int_0^{\infty} g(x,s)p(s,t)\mathrm{d}s, \tag{6}$$

where $g(x, s)$ is the PDF of $T(s)$ and $p(s, t)$ is a PDF of $S(t)$. The mean square displacement of the process $X(t)$ is therefore defined as [28]:

$$\left\langle X^2(t)\right\rangle = \int_0^{\infty} \left\langle T^2(s)\right\rangle p(s,t)\mathrm{d}s. \tag{7}$$

We have that:

$$\left\langle T^2(s)\right\rangle = s\alpha(\alpha-1)\lambda^{\alpha-2}. \tag{8}$$

Taking into account that the PDF of $S(t)$ is given by Ref. [28] we have:

$$p(s,t) = -\frac{\partial H(t,s)}{\partial s} = -\frac{\partial}{\partial s}\int_0^t h(t',s)\mathrm{d}t', \tag{9}$$

where $H(t, \tau)$ and $h(t, \tau)$ are the cumulative distribution function (CDF) and the PDF of $U(\tau)$ respectively. Repeating the calculations and applying the same reasoning as in Ref. [28], we obtain:

$$\left\langle X^2(t)\right\rangle = c\alpha(\alpha-1)\lambda^{\alpha-1}\int_0^{\infty} H(t,s)\mathrm{d}s. \tag{10}$$

Transforming the above to the Laplace domain $t \to u$ we have:

$$\int_0^{\infty} \mathrm{e}^{-ut}\left\langle X^2(t)\right\rangle \mathrm{d}t = \frac{c\alpha(\alpha-1)\lambda^{\alpha-2}}{u^{\gamma+1}}. \tag{11}$$

Transforming it back to the time domain t we obtain:

$$\left\langle X^2(t)\right\rangle = \frac{c\alpha(\alpha-1)\lambda^{\alpha-2}t^{\gamma}}{\Gamma(1+\gamma)}. \tag{12}$$

Since $0 < \gamma < 1$ then we obtain that the MSD of the subordinated process $X(t)$ behaves like in the subdiffusive case. In Fig. 1 one can observe that the MSD obtained from Monte Carlo simulations of the process $X(t)$ is in very good agreement with our theoretical result. The number of Monte Carlo repetitions was $n = 10^5$.

Now let us examine the probability $\lim_{b\to\infty} P(X(t) > b)$, where $X(t)$ is the subordinated tempered stable process defined in (3). Let us recall that for stable process $U(\tau)$ the classical result [29] states that $P(U(\tau) > b) \sim \tau C_{\alpha} b^{-\alpha}$ for large b and some constant C_{α}. Thus one can infer that for the tempered stable process there holds:

$$P(T(t) > b) \sim t\mathrm{e}^{-\lambda^{\alpha}t}C_{\alpha}^{\lambda}\mathrm{e}^{-\lambda b}b^{-\alpha}, \tag{13}$$

for some constant C_{α}^{λ}.

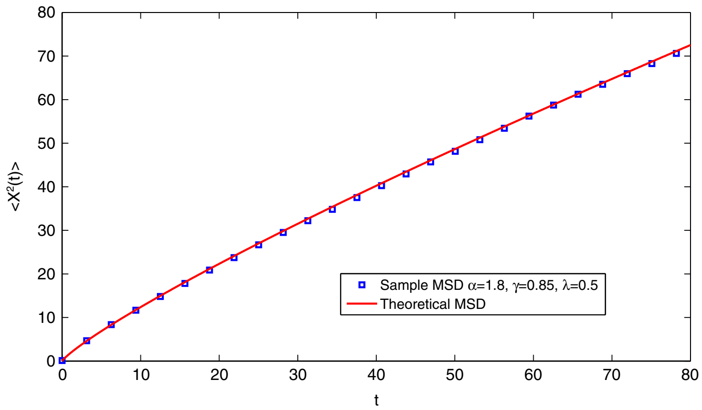

Fig. 1. Comparison of MSD obtained from Monte Carlo simulations of the process $X(t)$ and theoretical result given in (12). One can observe that MSD of the process $X(t)$ behaves as in the subdiffusive case. Parameters are: $\alpha = 1.8$, $\lambda = 0.5$, $\gamma = 0.85$.

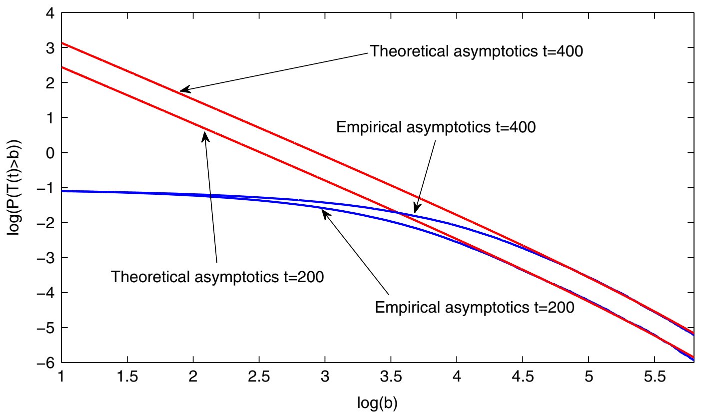

Fig. 2. Behavior of the asymptotics of $P(T(t) > b)$ as $b \to \infty$ obtained from Monte Carlo simulation of the process $T(t)$. One can observe that the right tail of the tempered stable process $T(t)$ behaves like $te^{-\lambda^\alpha t}C_\alpha^\lambda e^{-\lambda b}b^{-\alpha}$. For comparison we included results for two time points, namely $t = 200$ and $t = 400$. In each case one can observe that empirical results follow theoretical ones given in (13). Parameters are: $\alpha = 1.6$, $\lambda = 0.001$.

In Fig. 2 we present Monte Carlo results for asymptotics of the tempered stable process in comparison with theoretical results. One can observe that the tail of the tempered stable process behaves like $e^{-\lambda b}b^{-\alpha}$. The number of Monte Carlo repetitions was $n = 10^5$.

Now by using the asymptotic relation in (13) we formulate a similar expression for the subordinated process $X(t)$. Letting $b \to \infty$, we obtain:

$$\begin{aligned} P(X(t) > b) &= \int_0^\infty \int_b^\infty g(x,s)p(s,t)\mathrm{d}x\mathrm{d}s \\ &\sim C_\alpha^\lambda e^{-\lambda b}b^{-\alpha}\int_0^\infty e^{-\lambda^\alpha s}sp(s,t)\mathrm{d}s \\ &\sim C_\alpha^\lambda e^{-\lambda b}b^{-\alpha}\frac{-\partial}{\partial\lambda^\alpha}E_\gamma(-\lambda^\alpha t^\gamma) \\ &\sim C_\alpha^\lambda e^{-\lambda b}b^{-\alpha}t^\gamma E_{\gamma,1+\gamma}^2(-\lambda^\alpha t^\gamma). \end{aligned} \tag{14}$$

The function $E_{\alpha,\beta}^\rho(\cdot)$ is a generalized Mittag-Leffler function [30], defined as

$$E_{\alpha,\beta}^\rho(z) = \sum_{k=0}^\infty \frac{z^k(\rho)_k}{\Gamma(\alpha k+\beta)k!}, \quad (z,\beta\in\mathbb{C},\alpha>0),$$

where $(\rho)_k$ is the Pochhammer symbol [30]. The result presented in (14) follows from the differentiation rule of the Mittag-Leffler function [30]. In Fig. 3 we present Monte Carlo results for asymptotics of the subordinated tempered stable process with comparison with our theoretical result given in (14). Again, we observe almost perfect agreement between empirical and theoretical results.

Process $T(t)$ satisfies the following fractional Fokker–Planck equation [31]:

$$\frac{\partial g(x,t)}{\partial t} = \frac{\partial^{\alpha,\lambda}g(x,t)}{\partial x}, \tag{15}$$

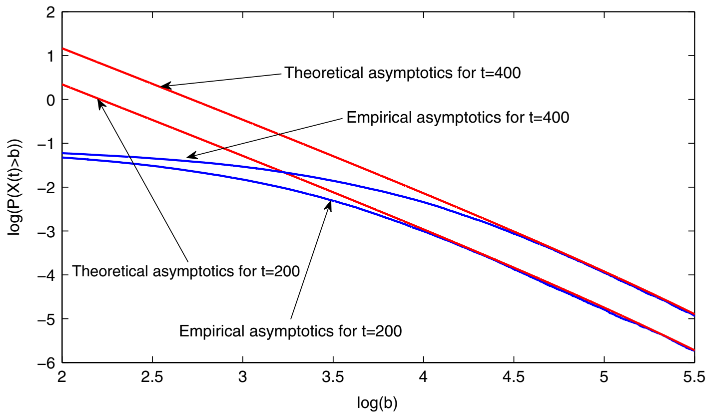

Fig. 3. Comparison of asymptotics of $P(X(t) > b)$ as $b \to \infty$ obtained from Monte Carlo simulations of the process $X(t)$ and the theoretical result given in (14). One can observe almost perfect agreement between empirical and theoretical results. For comparison we included results for two time points, namely $t = 200$ and $t = 400$. In each case one can observe that empirical results follow theoretical ones given in (14). Parameters are: $\alpha = 1.6$, $\gamma = 0.9$, $\lambda = 0.001$.

where the operator $\frac{\partial^{\alpha,\lambda}}{\partial x}$ is the inverse Fourier transform $\mathcal{F}^{-1}$ of (see Ref. [31])

$$\frac{\partial^{\alpha,\lambda} f(x)}{\partial x} = \begin{cases} \mathcal{F}^{-1}\Big(((\lambda + ik)^{\alpha} - \lambda^{\alpha} - ik\alpha\lambda^{\alpha-1})\hat{f}(k)\Big) & \text{for } \alpha \neq 1 \\ \mathcal{F}^{-1}\Big(\big((\lambda + ik)\log(\lambda + ik) - \lambda\log(\lambda) - ik(1 + \log(\lambda))\big)\hat{f}(k)\Big) & \text{for } \alpha = 1. \end{cases} \tag{16}$$

It is worth mentioning that in the case $\alpha \neq 1$ the above operator takes the following form:

$$\frac{\partial^{\alpha,\lambda} f(x)}{\partial x} = \mathrm{e}^{-\lambda x}\frac{\partial^{\alpha}}{\partial x}[\mathrm{e}^{\lambda x} f(x)] - \lambda^{\alpha} f(x) - \alpha\lambda^{\alpha-1}\frac{\partial f(x)}{\partial x},$$

where $\frac{\partial^{\alpha}}{\partial x}$ is the Riesz derivative.

Following the same technique as in Ref. [9] we derive the fractional counterpart of the Fokker–Planck equation for the process $X(t)$. Let us recall that the Laplace transform with respect to t of the process $S(t)$ has the form [9]:

$$\hat{p}(s, u) = u^{\gamma-1}\mathrm{e}^{-su^{\gamma}}.$$

From (6) we infer that Laplace transform $t \to u$ of the density of the process $X(t)$ satisfies the relation:

$$\hat{f}(x, u) = u^{\gamma-1}\hat{g}(x, u^{\gamma}) = \int_0^{\infty} g(x, s)\hat{p}(s, u)\mathrm{d}s. \tag{17}$$

In the Laplace domain equation (15) has the form:

$$u\hat{g}(x, u) - g(x, 0) = \frac{\partial^{\alpha,\lambda}\hat{g}(x, u)}{\partial x}. \tag{18}$$

Changing the variables $u \to u^{\gamma}$ in (18) and using the fact that $g(x, 0) = f(x, 0)$, we obtain:

$$u\hat{f}(x, u) - f(x, 0) = u^{1-\gamma}\frac{\partial^{\alpha,\lambda}\hat{f}(x, u)}{\partial x}. \tag{19}$$

Inversion of the above yields the fractional Fokker–Planck equation for the process $X(t)$:

$$\frac{\partial f(x, t)}{\partial t} = {}_0D_t^{1-\gamma}\frac{\partial^{\alpha,\lambda}}{\partial x} f(x, t). \tag{20}$$

The operator ${}_0D_t^{1-\gamma}$, $0 < \gamma < 1$ is the fractional Riemann–Liouville derivative, defined as:

$${}_0D_t^{1-\gamma} f(t) = \frac{1}{\Gamma(\gamma)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (t - s)^{\gamma-1} f(s)\mathrm{d}s.$$

Eq. (20) has a special form, namely it consists of two fractional operators. The first one is the time operator of the fractional Riemann–Liouville derivative connected with the inverse stable subordinator. The second one is the special operator connected with the external tempered stable process.

There is also an important link between the process $X(t)$ and the CTRW model. Recently in Ref. [32] the authors developed a triangular array scheme for the tempered α-stable process $T(t)$. Applying similar reasoning as in Ref. [33] (Theorem 3.1 and Remark 3.5) we are able to recover process $X(t)$ as a limit of property rescaled CTRW.

4. Testing and estimation procedures

In the first step of the estimation procedure, for realization of the subordinated tempered stable process with inverse stable subordinator, we divide the examined time series into two vectors. The first one (vector S) corresponds to lengths of constant time periods typical for subdiffusive processes while the second one (vector T) arises after removing constant time periods. Let us add that the length of the constant time period is equal to k if k consecutive observations have the same value. The first vector constitutes a sample of independent identically distributed (iid) random variables with the same distribution as the subordinator, while the second vector is related to the external process that in our case is tempered stable, i.e. after differencing one time we obtain here the iid sample from the tempered stable distribution.

This procedure was used in many cases for subordinated processes. For example, in case of the stable subordinator and stable external process the scheme was presented in Ref. [20] while for a tempered stable subordinator the procedure was proposed in Refs. [18,19].

The estimated parameter γ is the index of stability of the subordinator we can obtain it by using the method presented in Ref. [16], namely we fit the power function $x^{-\gamma}$ to the empirical tail of the vector S by using the least squares method. The other parameters, i.e. α and λ we can estimate from the increments of vector T by using for instance the method of moments. The estimators of the parameters related to the tempered stable distribution take the following forms:

$$\hat{\alpha} = \frac{6M_2^3 + 3M_3^2 - 2M_2M_4}{3M_2^3 + M_3^2 - M_2M_4},$$
$$\hat{\lambda} = -\frac{M_2(\hat{\alpha} - 2)}{M_3},$$

where $M_k = \frac{1}{n}\sum_{i=1}^{n}(T_i - \bar{T})^k$ for $k = 2, 3, 4$ and $\bar{T}$ is a sample mean. The other method that can be useful for estimation of the α and λ parameters is the maximum likelihood method. But because the tempered stable distribution the probability density function has no explicit form then in the considered case this technique requires numerical calculations and thus a long calculation time. Therefore in this paper we concentrate only on the mentioned method of moments.

In this paper we propose also to confirm our assumption of the distribution of the vector T, that is assumed to be tempered stable. There are many methods that can be useful in testing of the given distribution. One of the possible methods is based on the distance between theoretical and empirical cumulative distribution functions. The known statistical tests related to this methodology are for example Kolmogorov–Smirnov, Cramer–von Mises or Anderson–Darling. The complete description and applications of the mentioned tests one can find in Ref. [34]. Because of the tempered stable distribution the theoretical CDF is not given in the explicit form and approximation of this function requires numerical calculations then in this case we propose to use another approach based on the characteristic function. Let us mention that for the considered tempered stable distribution the characteristic function is given explicitly by formula (1). The testing method for a random sample $T_1, T_2, \ldots, T_n$ from the tempered stable distribution proceeds as follows:

1. Estimate the α and λ using for instance the method of moments.
2. Calculate the distance (square of the absolute value) between theoretical characteristic functions of the tempered stable distribution with estimated parameters and the empirical one calculated for a random sample. Let us mention the empirical characteristic function in point z for a given sample $T_1, T_2, \ldots, T_n$ takes the following form [35–37]:

$$\phi(z) = \frac{1}{n}\sum_{k=1}^{n} e^{ziT_k}.$$

3. Define the statistic W as a maximum of the distances for given arguments.
4. Generate the random sample from the tempered stable distribution with estimated values of parameters.
5. Calculate the maximum distance between the empirical (for a generated sample) and theoretical characteristic function (for a tempered stable distribution with estimated values of α, λ parameters). This value we denote as D_1.
6. Repeat points 4–5 N times and obtain $D_2, \ldots, D_N$. Small values of the statistic D_i are in favor of the null hypothesis that the data constitute a sample from the tempered distribution, however large ones mean its falsity. In order to see how unlikely such large values can happen within the assumption of the null hypothesis we calculate the p-value as a frequency of D_i values ($i = 1, 2, \ldots, N$) that exceed W.
7. If the p-value exceeds the given confidence level then we can suspect the data constitute a sample from the tempered stable distribution.

4.1. Simulation study

In order to show consistency of the estimators for parameters γ, α and λ obtained by using the methods mentioned above we use the bootstrap methodology. The bootstrap methodology is useful especially when the asymptotic properties of the estimators are not known. In our case the complicated formulas for the estimators make it impossible to determine their exact distribution. Therefore we concentrate on the bootstrap approach. In this paper we consider two types of bootstrap approach, namely parametric and nonparametric. The consistency will be proved by constructing the confidence interval

Table 1
Estimated values of γ, α and λ parameters from simulated samples from stable and tempered stable distribution (each of length 2500) and constructed confidence intervals for appropriate estimators calculated on the basis of the parametric and nonparametric bootstrap method.

Parameter	γ	α	λ
Theoretical value	0.8	1.8	0.5
Estimator	0.7944	1.8439	0.539
Parametric bootstrap confidence interval	[0.7579; 0.8529]	[1.5721; 1.8834]	[0.2107; 1.2999]
Nonparametric bootstrap confidence interval	[0.7484; 0.8791]	[1.5522; 1.9026]	[0.3251; 1.1182]

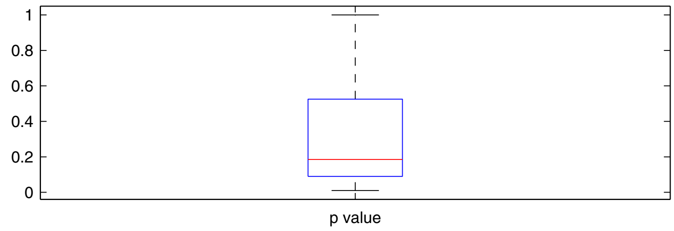

Fig. 4. Boxplot for p-value of test for tempered stable distribution. The result is presented for a simulated sample of length 2500 from tempered stable distribution with parameters $\alpha = 1.8$ and $\lambda = 0.5$. The obtained p-values are calculated on the basis of 100 simulations and 100 repetitions of the testing procedure.

for the appropriate estimator. In general, for a sample of independent observations $T_1, \ldots, T_n$ from a population with cumulative distribution function $F(x; \theta)$, the construction of the confidence interval for the estimator of unknown parameter θ proceeds as follows:

1. On the basis of sample $T_1, \ldots, T_n$ calculate the estimator for parameter θ, namely $\hat{\theta}$.
2. Generate the sample $T_1^{*1}, \ldots, T_n^{*1}$ from a distribution with CDF $F(x; \hat{\theta})$ (called a parametric bootstrap sample) and on the basis of this sample estimate $\hat{\theta}_1$.
3. Repeat point 2, N times and calculate $\hat{\theta}_1, \ldots, \hat{\theta}_N$.
4. On the basis of sample $\hat{\theta}_1, \ldots, \hat{\theta}_N$ (from the same distribution as $\hat{\theta}$) calculate the confidence interval for the given confidence level.

The above procedure, called parametric bootstrap, can be extended to the nonparametric case. Namely, instead of generating the sample from a distribution with CDF $F(x; \hat{\theta})$ we generate the bootstrap sample from a distribution from the empirical CDF calculated on the basis of the sample $T_1, \ldots, T_n$, [38–40].

In order to show consistency of the mentioned estimation methods we generate two iid samples from the stable (with index of stability $\gamma = 0.8$) and tempered stable distribution (with parameters $\alpha = 1.8$ and $\lambda = 0.5$). Each sample consists of 2500 elements. Then, we estimate the unknown parameters and on the basis of the bootstrap methodology we construct confidence intervals in the parametric and nonparametric case for the confidence level 0.05. For the analysis we take $N = 1000$. In Table 1 we present the obtained results.

As we observe, the obtained estimates are very close to the theoretical values of the appropriate parameters.

The mentioned testing procedure for the tempered stable distribution we also present for a simulated sample. The simulated vector constitutes iid random variables from the tempered stable distribution with parameters $\alpha = 1.8$ and $\lambda = 0.5$. In Fig. 4 we present the boxplot for p-values obtained for 100 repetitions of simulations. In the considered case we analyze a vector of 2500 elements, and take $N = 100$.

5. Real data analysis

In order to illustrate the theoretical results in this section we examine financial data from the Polish market, namely the 3 months WIBID rate (Warsaw interbank bid rate) quoted daily for the period 2.01.1995–26.09.2012 (4456 observations). The WIBID is the interest rate of the Warsaw Interbank Market for deposits taken by banks in PLN for a specified period (in our case 3 months), announced by the Reuters service on its web site at 11.00 on the specified quotation date. As the Polish market is relatively young one can expect that subdiffusive behavior is more likely to be observable compared to more developed world exchanges. For the analysis we take the highest daily price (with accuracy 0.01) from the mentioned period. In Fig. 5 we present the examined time series. We observe here the time periods when the time series takes the same value that is typical for subdiffusive processes.

In the first step of our analysis, according to the procedure presented in the previous section, we divide the examined time series into two vectors. The first one represents lengths of constant time periods (vector S), while the second — the process that arises after removing the constant time periods (vector T). According to our assumption the vector S consists of iid random variables from the γ-stable distribution with the index of stability $\gamma < 1$ while the vector T is a realization of the tem-

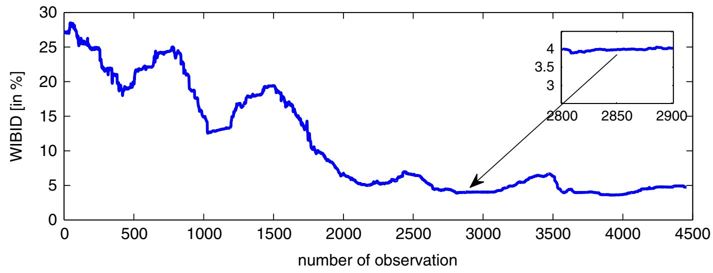

Fig. 5. The examined time series that describes the highest daily (with accuracy 0.01) WIBID rate from the period 2.01.1995–26.09.2012. We observe here the constant time periods typical for subdiffusive processes.

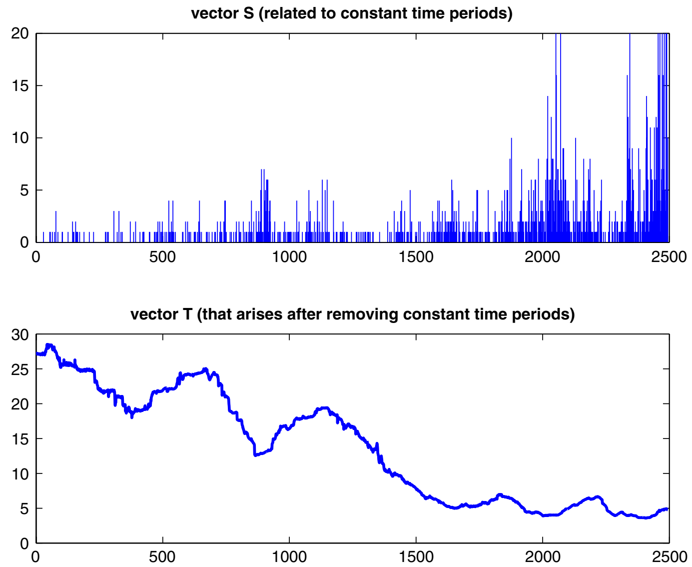

Fig. 6. The constant time periods (vector S) and model related to the external process (vector T).

pered stable process; thus after differencing it consists of iid random variables from the tempered stable distribution. In Fig. 6 (top panel) we present lengths of constant time periods as well as the vector that arises after removing traps (bottom panel).

In order to estimate the γ parameter we use the mentioned method based on the tail behavior for a stable distribution, see Section 4. For the given real data set we obtain $\hat{\gamma} = 0.79$. Next we can analyze the external process. According to our assumption, the increments of the vector T consist of iid random variables from the tempered α-stable distribution. At the beginning we should confirm that the tempered stable distribution is a better choice than the other known distributions, for example stable. In order to confirm the assumption of the distribution we examine the empirical tail of the increments of vector T. In Fig. 7 we present in log–log scale the empirical tail and estimated power function (corresponding to a stable law) and truncated power function — $e^{-\lambda x}x^{-\alpha}$ (corresponding to the tempered stable law). As we observe, the fitted truncated power function approximates in a better way the empirical tail. In order to confirm the assumption of tempered stable distribution we also use the test mentioned in the previous section which is based on the distance between empirical and theoretical characteristic functions. As a result we obtain a p-value equal to 0.17 (for 100 repetitions of the estimation procedure) that confirms the hypothesis that the tempered stable distribution cannot be rejected for the confidence level 0.05. At the end we estimate the parameters corresponding to the tempered stable distribution. In this case we use the method of moments presented in Section 4. As a result we obtain: $\hat{\alpha} = 1.8738$, $\hat{\lambda} = 0.5264$.

6. Conclusions

In this paper we analyzed the tempered stable process with α-stable waiting-times. We classified it as a subdiffusive model and derived its tail behavior. We presented in detail the estimation and testing procedure for the proposed system. Moreover we calibrated the subordinated tempered stable process to a data set of Warsaw interbank bid rates.

The proposed approach can serve as an alternative to existing models. It reflects not only the tempered stable behavior of the analyzed time series but also allows for modeling the constant periods visible in the data. We also should mention here that the subordinated process might be useful as a model of high-frequency (especially tick–tick) data, when the waiting-times between the individual transactions are observed.

We believe that the methodology proposed here will serve as a useful tool in data modeling.

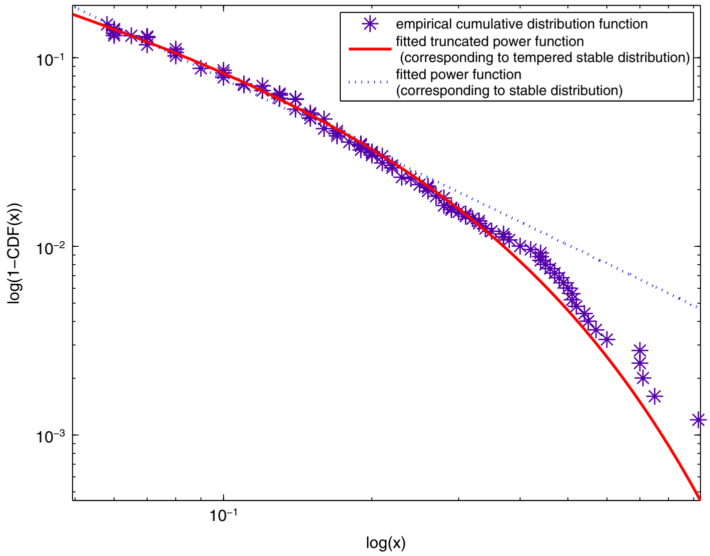

Fig. 7. The empirical cumulative distribution function with fitted power function (corresponding to stable law) and truncated power function (corresponding to tempered stable law) in log–log scale.

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