Physica A 388 (2009) 407–418

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Coupled continuous-time random walk approach to the Rachev–Rüschendorf model for financial data

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ARTICLE INFO

Article history:
Received 4 July 2008
Received in revised form 13 October 2008
Available online 5 November 2008

PACS:
89.65.Gh
05.40.Fb

Keywords:
Continuous-time random walk
Log-returns of financial instruments
Black–Scholes formula
Cox–Ross–Rubinstein model
Randomization
Alternative model

ABSTRACT

In this paper we expand the Rachev–Rüschendorf asset-pricing model introducing a coupled continuous-time-random-walk-(CTRW)-like form of the random number of price changes. Such a form results from the concept of the random clustering procedure (that resembles the coarse-graining methods of statistical physics) and, on the other hand, indicates applicability of the CTRW idea, widely used in physics to model anomalous diffusion, for describing financial markets. In the framework of the proposed model we derive the limiting distributions of log-returns and the corresponding pricing formulas for European call option. In order to illustrate the obtained theoretical results we present their fitting with several sets of financial data.

1. Introduction

In physics, the notion of continuous-time random walk[1] (CTRW for short) is commonly accepted as a powerful mathematical tool for analysis of relaxation and transport phenomena in complex systems that exhibit anomalous dynamical behavior resulting in a non-Gaussian asymptotic distribution of the diffusion front [1–14]. CTRW approach generalizes attempts to explain the observed departure from the normal (Gaussian) distribution based on the theory of Lévy-stable distributions. The alternative models obtained this way lead to scale mixtures of Lévy-stable laws (including the Gaussian law as a special case), where the mixing distribution depends on the detailed properties of the considered CTRW process [5,15,16]. For decoupled CTRW's[2] the mixing distribution is also stable and leads to the fractional stable limiting law [16–18]. Otherwise, more complicated mixing distributions may appear, connected not only with the stable but also with a generalized arcsine distribution (i.e. the beta distribution with parameters p and $q = 1 - p$ for some $0 < p < 1$) [16,19,20].

Recently, CTRW's have been applied also in finance to model the evolution of log-prices, the risk process, or the forward rate dynamics [20–24]. Distributions of returns for real prices of financial instruments are the fundamental components in the portfolio security [25]. On the one hand, models of the capital market equilibrium study the structure of asset prices and

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1 i.e. a walk with random waiting times between successive random jumps.
2 i.e. when the waiting times are independent of the jumps.

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doi:10.1016/j.physa.2008.10.041

its connection with empirical anomalies. On the other hand, option pricing models show empirical biases as the strike price bias or the smile effect, see Ref. [26].

Options have been traded for centuries but they remained relatively obscure financial instruments until the introduction of a listed option exchange in 1973. Since then option trading has enjoyed an expansion unprecedented on the US security markets. The modern option pricing theory begins in 1973. At that time F. Black and M. Scholes presented the first completely satisfactory equilibrium option pricing model taking into consideration the real aspect of financial markets [27]. In the same year, R. Merton extended their model using random interest rates [28], see also Refs. [29,30]. These path-breaking articles have formed the basis for many subsequent academic studies.

Option pricing theory is relevant to almost every area of finance. For example, virtually all corporate securities can be interpreted as portfolios of put and call options on the assets of the firm. Indeed, the theory applies to a very general class of economic problems; namely, the valuation of contracts where the outcome to each party depends on a quantifiable uncertain future event. Unfortunately, the mathematical tools employed in the Black–Scholes and Merton models are quite advanced and have tended to obscure the underlying economics [25,31].

The idea of looking at a binomial model as a discrete-time approximation to continuous-time diffusion were initially justified by J.C. Cox, J.E. Ross and M. Rubinstein in 1979 when they presented a simple discrete-time option pricing model (the CRR model, for short) [31,32]. This approach (categorized as a Lattice Model or Tree Model because of the graphical representation of the stock price over the large number of intervals or steps) includes the Black–Scholes model as a limiting case. The authors developed the proposed CRR model for a call option on a stock that pays no dividends. Also, they showed exactly how the model can be used to lock in pure arbitrage profits if the market price of an option differs from the value given by the model. By taking the limits in a different way, they obtained the Cox–Ross jump process model (1976) as another special case [33].

The Gaussian distribution of log-returns, appearing in the Black–Scholes model or as a limiting law in the CRR approach, is often observed being far from the empirical distributions of financial data [35]. Seeking more general and more realistic limiting models, in 1994 S.T. Rachev and L. Rüschendorf extended the CRR model by introducing two additional randomizations in the binomial price model: a randomization of the number of price changes and a randomization of the ups and downs in the price process [34,35]. As a result, they obtained the Rachev–Rüschendorff price models (the RR models, for short) with fat tails, higher peaks in the center and non-symmetric distribution, usually observed in typical asset return data [36].

In this paper, following the methodology proposed in Ref. [24], we expand the RR asset-pricing model introducing a coupled CTRW-like form for the random number of price changes. The proposed methods of randomization of the number of price jumps are based on the notion of random coarse graining transformation of CTRW's, introduced and studied in Refs. [5,16]; and they assume some agglutination of the successive price ups and downs in the groups (clusters) of random sizes. Such an approach provides quite realistic description of the asset price on the real market where indeed we observe randomizations of the value of successive groups of jumps. The considered form of the random number of price changes reveals significance of a new idea of CTRW's in the double-array limit scheme [18] in modeling of financial markets.

The article is structured as follows: Section 2 contains a brief survey on the idea and basic properties of the RR model. In Section 3 two clustering schemes for constructing the random number of price changes in the RR model (connected with different hedging strategies) are proposed. Then, in Section 4, the limiting distributions of log-returns and the corresponding pricing formulas for European call option (going beyond the classical Black–Scholes formula) are derived in the framework of the considered model. Finally, in Section 5 the third model, which combines properties of both previous schemes, is proposed and studied. Moreover, the obtained theoretical results are applied for analysis of several sets of financial data. Concluding remarks are given in the last section.

2. The Rachev–Rüschendorf model for financial market

The Rachev–Rüschendorf (RR) model is based on the classical CRR binomial model of market formed by a bank account $(B_k)_{k\geq 1}$ and some stock of value $(S_k)_{k\geq 1}$ [37]. In the CRR model one assumes that in the time interval $[0, T]$ for a fixed time T (with 1 year as the time unit) both the value of bank account and the stock price change n times at instants of time $k\frac{T}{n}$, $k = 1, \ldots n$; and the evolution is given by the following rules:

$$B_k = \Lambda B_{k-1} = \Lambda^k B_0, \tag{2.1}$$

where $\Lambda = 1 + r\frac{T}{n}$ for the constant 1-year interest rate $r > 0$, while

$$S_k = u^{\epsilon_{n,k}} d^{1-\epsilon_{n,k}} S_{k-1} = S_0 \prod_{i=1}^{k} u^{\epsilon_{n,i}} d^{1-\epsilon_{n,i}}, \tag{2.2}$$

where for each n the parameters $u = u_n > 1$ and $0 < d = d_n < 1$ correspond to jump up or down, respectively, of the stock price; and $(\epsilon_{n,k})_{k\geq 1}$ is a sequence of independent and identically distributed (i.i.d.) random variables such that $\Pr(\epsilon_{n,k} = 1) = p_n = 1 - \Pr(\epsilon_{n,k} = 0)$ for some probability $0 < p_n < 1$ of jumping up. The initial values B_0 and S_0 are positive constants.

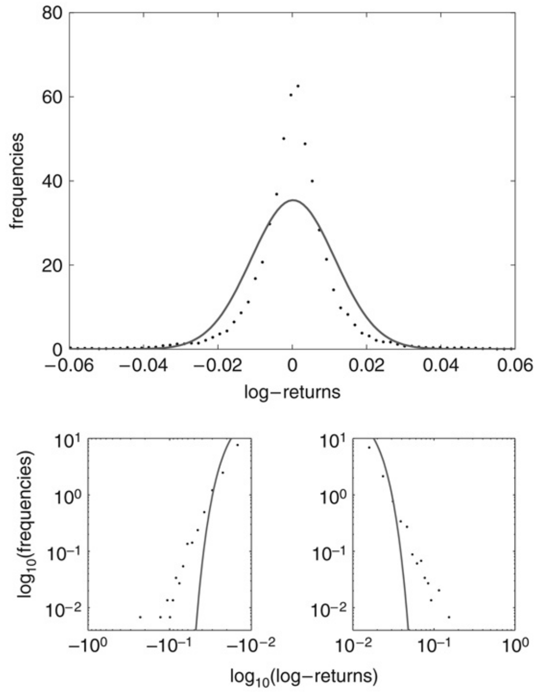

Fig. 1. Top panel: Histogram of daily log-returns of Standard & Poor's 500 index [42] compared with the fitted Gaussian probability density function. Bottom panel: The corresponding left and right boundary data ranges in the log–log scale. For the fit parameters, see Table 1.

Observe that at time T, as a consequence of (2.2), the stock price $S(T) = S_n$ satisfies

$$\log\left(\frac{S(T)}{S_0}\right) = \sum_{k=1}^{N_n} X_{n,k}, \tag{2.3}$$

where $N_n = n$ and $X_{n,k} = U_n\epsilon_{n,k} + D_n(1-\epsilon_{n,k})$ (where $U_n = \log u_n, D_n = \log d_n$) so that $(X_{n,k})_{k\geq 1}$ is a sequence of i.i.d. random variables for which $\Pr(X_{n,k} = U_n) = 1 - \Pr(X_{n,k} = D_n) = p_n$. In other words, the log-return $\log(S(T)/S_0)$ at time T results from $N_n = n$ i.i.d. price jumps represented by $X_{n,k}$, $k \geq 1$.

If we take $u_n = \mathrm{e}^{\tilde{\sigma}\sqrt{\frac{T}{n}}}$, $d_n = \mathrm{e}^{-\tilde{\sigma}\sqrt{\frac{T}{n}}}$ for volatility $\tilde{\sigma} > 0$ such that $2r < \tilde{\sigma}^2$, and for $p_n = \frac{1}{2} + \frac{\alpha}{2\tilde{\sigma}}\sqrt{\frac{T}{n}}$ for some $\alpha \in \mathbb{R}$, then for large n (i.e. for the large number of stock-price changes) we obtain [35,37–41] asymptotically normal distribution $\mathcal{N}(\alpha T, \tilde{\sigma}^2 T)$ of log-returns $\log(S(T)/S_0)$ and, as a consequence, the famous Black–Scholes formula

$$C_0(T) = S_0\Phi\left(\frac{\log(S_0/K) + \left(r + \frac{1}{2}\tilde{\sigma}^2\right)T}{\tilde{\sigma}\sqrt{T}}\right) - K\mathrm{e}^{-rT}\Phi\left(\frac{\log(S_0/K) + \left(r - \frac{1}{2}\tilde{\sigma}^2\right)T}{\tilde{\sigma}\sqrt{T}}\right) \tag{2.4}$$

for the rational price $C_0(T)$ of a standard European call option with the maturity time T and the strike price K, where $\Phi(\cdot)$ stands for the standard normal distribution function. (Observe that parameter α controls the drift in the Brownian motion underlying the Black-Sholes formula.) However, it is commonly observed that the Gaussian distribution of log-returns, appearing in the CRR and the Black–Scholes models, fails to demonstrate a good fit with the empirical distributions of financial data [35] (see e.g. Fig. 1 where the histogram of daily log-returns of Standard & Poor's 500 index from years 1950–2005 [42] is compared with the fitted Gaussian law). Hence there is a need of looking for new models which would better describe the real market data.

The classical results provided by the CRR model can be extended to more general form in the framework of the RR model that generalizes the CRR idea considering random number N_n of price changes in (2.3). In the RR model one takes into account N_n that is independent of sequence $(X_{n,k})_{k\geq 1}$, representing the price jumps. Moreover, one assumes that the limiting distribution of the ratio $\frac{N_n}{n}$ for large n is provided by some nonnegative random variable Y so that

$$\Pr\left(\frac{N_n}{n} \leq y\right) \underset{n\to\infty}{\longrightarrow} \Pr(Y \leq y) = F_Y(y) \tag{2.5}$$

for any y being the continuity point of F_Y, the distribution function of Y. In such a case one obtains [34] that the limiting distribution of log-returns $\log(S(T)/S_0)$ is a mixture of F_Y and the Gaussian distribution, related to the random variable

$$Z_T = \tilde{\sigma}\sqrt{T}\sqrt{Y}X_0 + \alpha TY, \tag{2.6}$$

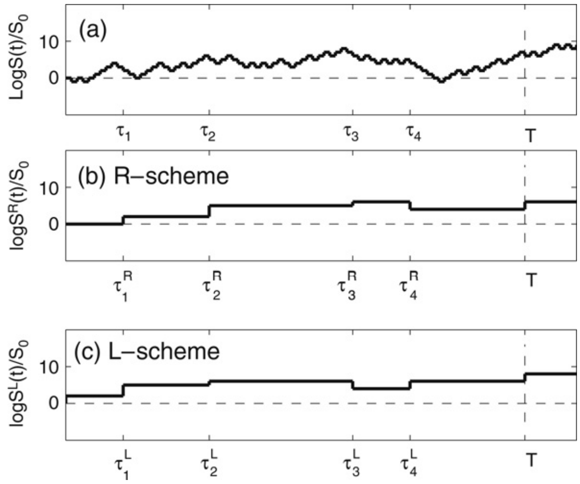

Fig. 2. An exemplary realization of the clustering schemes: (a) the initial CRR log-returns process (2.3); (b) log-returns (3.10) in the R-scheme; (c) log-returns (3.12) in the L-scheme.

where X_0 has the standard normal distribution $\mathcal{N}(0, 1)$ and is independent of Y. The corresponding limiting rational price C_0 of a European call option with the maturity time T and the strike price K is then equal to [34]

$$C_0(T) = S_0 P\left(\tilde{\sigma}\sqrt{T}\sqrt{Y}X_0 + \left(r + \frac{1}{2}\tilde{\sigma}^2\right)TY \geq \log(K/S_0)\right)$$
$$- K\mathrm{E}e^{-rTY} P\left(\tilde{\sigma}\sqrt{T}\sqrt{Y_T}X_0 + \left(r - \frac{1}{2}\tilde{\sigma}^2\right)TY_T \geq \log(K/S_0)\right), \tag{2.7}$$

where Y_T is a random variable independent of X_0 that has the Laplace transform of the form

$$\psi_{Y_T}(\theta) = \frac{\psi_Y(rT+\theta)}{\psi_Y(rT)}. \tag{2.8}$$

(Here $\psi_Y(\cdot)$ denotes the Laplace transform of Y).

In the next sections we shall propose special, random clustering schemes for constructing N_n, the number of jumps in the RR model. For the introduced forms of N_n we derive the explicit formulas for the limiting distributions of log-returns and for the limiting call option price. Then we compare the obtained theoretical results with financial data.

3. Clustering schemes for the RR model

In the CRR model the log-return process can be seen as a simple CTRW with jumps $X_{n,k}$ performed at instants of time $k\frac{T}{n}$, $k \geq 1$. In order to randomize the number of stock price changes let us apply to such a CTRW the random coarse graining transformation, introduced to model relaxation phenomena [9,16], that roughly speaking, means clustering jumps in some "packets" of random sizes. More precisely, we take $(M_j)_{j\geq 1}$, a sequence of i.i.d. random indices[3], independent of $\left(X_{k,n}\right)_{k,n\geq 1}$. Then, by means of (M_j), we choose randomly the instants of time $\tau_i = \mu_i\frac{T}{n}$, where $\mu_0 = 0$, $\mu_i = \sum_{j=1}^{i} M_j$ for $i \geq 1$. The successive stock-price ups and downs performed (in the CRR model) after time τ_{i-1} and before or at time τ_i are represented by the cumulative jump

$$\tilde{X}_{n,i} = \sum_{k=\mu_{i-1}+1}^{\mu_i} X_{n,k}. \tag{3.9}$$

The random variable M_i corresponds hence to the size of the ith packet of the price changes, and $\tilde{X}_{n,i}$ is the cumulative price change representing this packet.

The time $\tilde{\tau}_i$ in the interval $\langle\tau_{i-1}, \tau_i\rangle$ when the cumulative jump $\tilde{X}_{n,k}$ is performed in the transformed CTRW can be chosen in different ways. Here we consider two possibilities (see Fig. 2):

1. $\tilde{\tau}_i = \tilde{\tau}_i^R = \tau_i$, i.e. the cumulative jump performed at the right end of the corresponding time interval;
2. $\tilde{\tau}_i = \tilde{\tau}_i^L = \tau_{i-1}$, i.e. the cumulative jump performed at the left end of the interval.

[3] i.e. positive integer-valued random variables.

The two proposed ways of choosing $\tilde{\tau}_i$ lead to two schemes (R-scheme, L-scheme, respectively) for constructing the number of price changes in the RR model, and on the other hand, they are connected with different hedging strategies represented by the standard European call option. The idea of using the R-scheme ($\tilde{\tau}_i^R = \tau_i$) has been widely discussed in Ref. [24]. Here we remind that the R-scheme is adequate to the case when we do not observe the asset price at any available time point but monitor the value at the randomly chosen instants of time. Hence, to evaluate the option price on time T we take into account only information until the latest of the time instants τ_i before or at T, see Fig. 2(b). For such a strategy, the resulting log-return at time T is given by:

$$\log\left(\frac{S^R(T)}{S_0}\right) = \sum_{k=1}^{K_n^R} \tilde{X}_{n,k}, \tag{3.10}$$

where

$$K_n^R = \max\{k \in \mathbb{N} : \tau_k \le T\} = \max\{k \in \mathbb{N} : \mu_k \le n\}. \tag{3.11}$$

Hence, the log-returns are expressed by means of a coupled CTRW process resulting from the waiting times $\tau_i - \tau_{i-1} = M_i \frac{T}{n}$ for the cumulative jumps $\tilde{X}_{n,i}$, (Eq. (3.9)). Since we have $\tilde{X}_{n,i} = \sum_{k=1}^{M_i} X_{n,k+\mu_{i-1}}$, the coupling (alternative to the one studied in Refs. [19,20]) is given here by the random block size M_i that provides stochastic dependence between the waiting time and the corresponding cumulative price jump.

In the L-scheme ($\tilde{\tau}_i^L = \tau_{i-1}$) we take into account different information than in the previous case, see Fig. 2(c). Namely, at the beginning of the considered time period we predict not only its length (by means of M_i), but also the values of the CRR price changes (instead of waiting for them until the end of the period). Hence, we predict the asset price at time T by means of information until the first time τ_i after T, forecasting the future. The log-returns at time T are then given by the following coupled CTRW-like form:

$$\log\left(\frac{S^L(T)}{S_0}\right) = \sum_{k=1}^{K_n^L} \tilde{X}_{n,k}, \tag{3.12}$$

where

$$K_n^L = \min\{k \in \mathbb{N} : \tau_k > T\} = \min\{k \in \mathbb{N} : \mu_k > n\} = K_n^R + 1, \tag{3.13}$$

in analogy to (3.10) and (3.11). However, in this case $\tau_i - \tau_{i-1} = M_i \frac{T}{n}$ should be interpreted as the residence time after the cumulative jump $\tilde{X}_{n,i}$ has occured. As we present in the next section, this modification can lead us to a new result.

Observe that formulas (3.10) and (3.12) can be rewritten into the RR-model forms

$$\log\left(\frac{S^R(T)}{S_0}\right) = \sum_{k=1}^{N_n^R} X_{n,k}, \tag{3.14}$$

$$\log\left(\frac{S^L(T)}{S_0}\right) = \sum_{k=1}^{N_n^L} X_{n,k}, \tag{3.15}$$

where the corresponding random numbers of price changes read as

$$N_n^R = \sum_{j=1}^{K_n^R} M_j = \mu_{K_n^R} \quad \text{and} \quad N_n^L = \sum_{j=1}^{K_n^L} M_j = \mu_{K_n^L}.$$

The RR-model forms (3.14) and (3.15) allow us to derive limiting distributions of the log-returns at time T in R- and L-schemes by means of the general result for the RR-model [34].

4. Log-returns limiting distributions for the clustering schemes

Applying the general results for the RR model [34] (reminded in Section 2) and the well-known facts of the renewal theory [16,43,44] we derive the log-returns limiting distributions and the resulting formulas for the rational price of the European call option in the framework of the proposed R- and L-schemes for constructing the number of price jumps. (The derivations are similar to those presented in Ref. [16].)

If $\mathrm{E}M_j < \infty$, i.e. the random packet sizes (or equivalently the waiting/residence times $\tau_i - \tau_{i-1}$) are of finite mean value, then both $\frac{N_n^R}{n}$ and $\frac{N_n^L}{n}$ tend to 1 with probability 1 as $n \to \infty$ [44]; and as a consequence none of the proposed schemes goes beyond the results derived in the framework of the CRR model; in particular, beyond the Black–Scholes pricing formula (2.4). On the contrary, some interesting extensions of the classical results can be obtained if we take into account heavy-tailed random packet sizes (or equivalently the waiting/residence times $\tau_i - \tau_{i-1}$). Namely, in the case when for some parameter

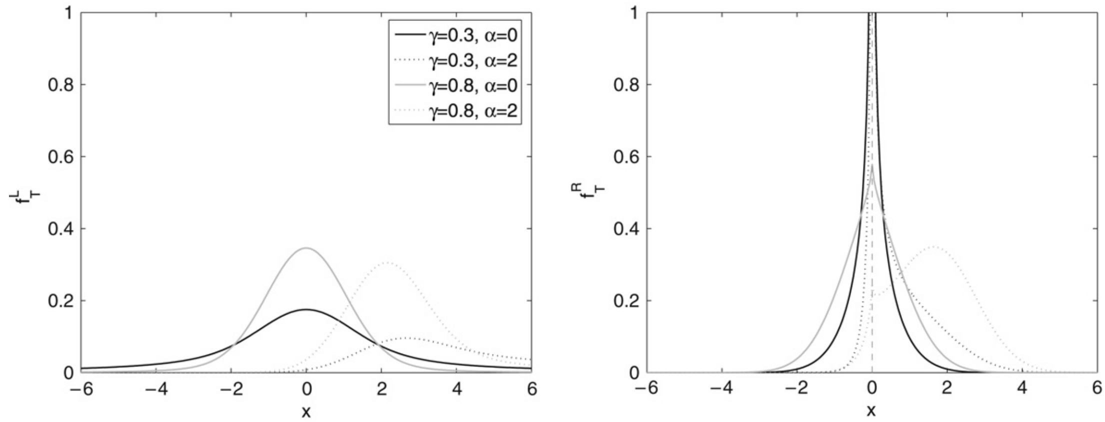

Fig. 3. Exemplary plots of the log-returns limiting probability density functions in the R- and L-schemes ($\tilde{\sigma} = 1, T = 1$). Right panel: $f_T^R(x; \alpha, \gamma)$, Eq. (4.17). Left panel: $f_T^L(x; \alpha, \gamma)$, Eq. (4.19).

$0 < \gamma < 1$ we have

$$\Pr(M_1 > m) \sim m^{-\gamma} \quad \text{for } m \to \infty, \tag{4.16}$$

one can derive nondegenerate limiting distributions of $\frac{N_n^R}{n}$ and of $\frac{N_n^L}{n}$ for large n. Since by definition $N_n^R \le n$ while $N_n^L > n$, these distributions have to be different. Indeed, in the R-scheme one obtains the generalized arcsine limiting law with parameter $p = \gamma$, represented by a random variable $\mathcal{B}_\gamma$, while the distribution of $\mathcal{B}_\gamma^{-1}$ appears as a limit in the L-scheme [16, 43].

Due to this difference in the asymptotic behavior of the numbers of price changes, the considered clustering schemes with heavy-tailed packet sizes yield different asymptotic behavior of the log-returns. Namely, for the R-scheme with the packet sizes satisfying (4.16), the limiting distribution of the log-returns $\log(S^R(T)/S_0)$ as $n \to \infty$ is related to the random variable Z_T, Eq. (2.6), with $Y = \mathcal{B}_\gamma$. Its probability density function reads as

$$f_T^R(x; \alpha, \gamma) = \frac{\sin(\pi\gamma)}{\sqrt{2\pi^3\tilde{\sigma}^2 T}} \int_0^1 \mathrm{e}^{-\frac{(x-\alpha T y)^2}{2y\tilde{\sigma}^2 T}} \frac{y^{\gamma-3/2}\mathrm{d}y}{(1-y)^\gamma}, \tag{4.17}$$

and the resulting limiting European call option price (with the maturity time T and the strike price K) is given by the formula

$$C_0^R(T; \gamma) = S_0 \frac{\sin(\pi\gamma)}{\pi} \int_0^1 \Phi\left(\frac{\log(S_0/K) + \left(r + \frac{1}{2}\tilde{\sigma}^2\right) Ty}{\tilde{\sigma}\sqrt{Ty}}\right) \frac{y^{\gamma-1}\mathrm{d}y}{(1-y)^\gamma}$$
$$- K \frac{\sin(\pi\gamma)}{\pi} \int_0^1 \mathrm{e}^{-rTy} \Phi\left(\frac{\log(S_0/K) + \left(r - \frac{1}{2}\tilde{\sigma}^2\right) Ty}{\tilde{\sigma}\sqrt{Ty}}\right) \frac{y^{\gamma-1}\mathrm{d}y}{(1-y)^\gamma}, \tag{4.18}$$

see also Ref. [24]. On the contrary, in the framework of the L-scheme and assuming again (4.16) one obtains that the log-returns limiting distribution of $\log(S^L(T)/S_0)$ for $n \to \infty$ is described by the random variable Z_T, Eq. (2.6), with $Y = \mathcal{B}_\gamma^{-1}$. Its probability density function is as follows

$$f_T^L(x; \alpha, \gamma) = \frac{\sin(\pi\gamma)}{\sqrt{2\pi^3\tilde{\sigma}^2 T}} \int_1^\infty \mathrm{e}^{-\frac{(x-\alpha T y)^2}{2y\tilde{\sigma}^2 T}} \frac{y^{-3/2}\mathrm{d}y}{(y-1)^\gamma}, \tag{4.19}$$

and the pricing formula takes the form

$$C_0^L(T; \gamma) = S_0 \frac{\sin(\pi\gamma)}{\pi} \int_1^\infty \Phi\left(\frac{\log(S_0/K) + \left(r + \frac{1}{2}\tilde{\sigma}^2\right) Ty}{\tilde{\sigma}\sqrt{Ty}}\right) \frac{y^{-1}\mathrm{d}y}{(y-1)^\gamma}$$
$$- K \frac{\sin(\pi\gamma)}{\pi} \int_1^\infty \mathrm{e}^{-rTy} \Phi\left(\frac{\log(S_0/K) + \left(r - \frac{1}{2}\tilde{\sigma}^2\right) Ty}{\tilde{\sigma}\sqrt{Ty}}\right) \frac{y^{-1}\mathrm{d}y}{(y-1)^\gamma}. \tag{4.20}$$

Notice that formulas (4.17) and (4.19) for the log-return limiting probability density function are alternative to the function resulting from the normal coupling, considered in Ref. [20].

Observe that both, the R- and L-scheme log-returns limiting distributions f_T^R, Eq. (4.17), and f_T^L, Eq. (4.19), are some mixtures of the Gaussian law and a generalized arcsine distribution. However, there is an essential difference between them and, as a consequence, between the pricing formulas obtained in the framework of the R- and L-schemes. In Fig. 3

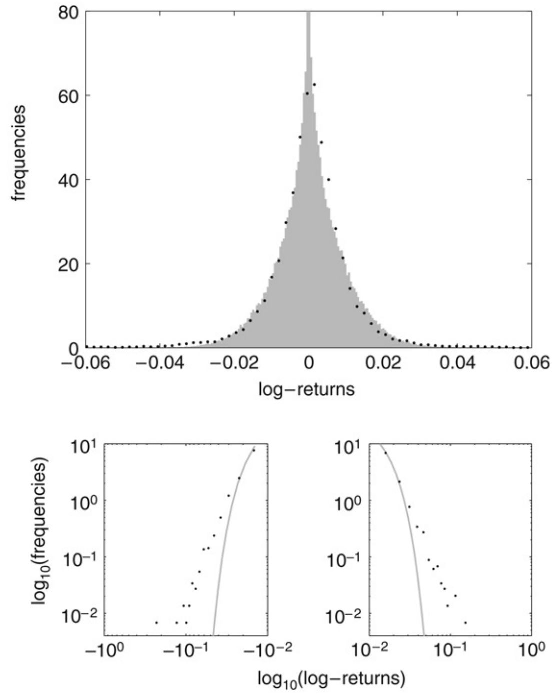

Fig. 4. Top panel: Histogram of daily log-returns of Standard & Poor's 500 index [42] (black) fitted with the R-scheme log-returns limiting distribution f_T^R, Eq. (4.17), (grey). Bottom panel: The corresponding left and right boundary data ranges in the log–log scale. For the fit parameters, see Table 1.

Table 1
Parameters of the log-returns limiting distributions (obtained in the CRR model and in the proposed clustering schemes) fitted with the empirical distribution of daily log-returns of Standard & Poor's 500 index [42].

Model	α		$\tilde{\sigma}$		
CRR	0		0.216		
Scheme	γ		α	$\tilde{\sigma}$	
R	0.55		0.292	0.2293	
L	0.72		0.0365	0.1146	
Scheme	γ^R	γ^L	α	$\tilde{\sigma}$	p_0
R/L	0.5	0.68	0.0365	0.2102	0.92

there are shown exemplary plots of the probability density functions $f_T^R(x;\alpha,\gamma)$ and $f_T^L(x;\alpha,\gamma)$ for different values of parameters α, γ and T. Note that for both functions $\alpha = 0$ gives a symmetric log-returns probability distribution, while $\alpha \neq 0$ provides a skewed limiting law. Moreover, for the R-scheme one can observe different behaviors of $f_T^R(x;\alpha,\gamma)$ at point $x = 0$ for $\gamma \leq 1/2$, when the considered function is not defined at zero but has an asymptote at this point, and for $\gamma > 1/2$, when zero is a continuity point. Another difference lies in the fact that for the R-scheme one always gets finite mean value $\int_{-\infty}^{\infty} x f_T^R(x;\alpha,\gamma)\mathrm{d}x$ (that indicates light tail behavior). In contrary, for the L-scheme the mean value $\int_{-\infty}^{\infty} x f_T^L(x;\alpha,\gamma)\mathrm{d}x$ is finite only for $\alpha = 0$, $\gamma > 1/2$.

To show applicability for analysis of financial data of formulas (4.17) and (4.19), obtained in the framework of the R and L-schemes as the log-returns limiting distributions, we have fitted functions f_T^R and f_T^L with the empirical distribution of daily log-returns of Standard & Poor's 500 index [42] (i.e. the data presented already in Fig. 1 in the context of the Black–Scholes and the CRR models). For detailed investigations of the quality of fits we have used the Kolmogorov statistics. The limiting distributions have been simulated by means of the Monte Carlo technique. The results are demonstrated in Figs. 4 and 5. The obtained values of the parameters are collected in Table 1. Let us remind that parameter $\tilde{\sigma} > 0$ represents the volatility, α controls the skewness of the log-returns limiting distributions, and $0 < \gamma < 1$ is the heavy-tail power-law exponent of the random waiting/residence times between the cumulative price jumps.

5. R/L-scheme and its application to financial data analysis

Analysis of the fitting quality for the log-returns limiting distributions resulting from the R-scheme, the L-scheme and the CRR model show that the use of the Black–Scholes model, a limiting case of the CRR approach, gives worse results than

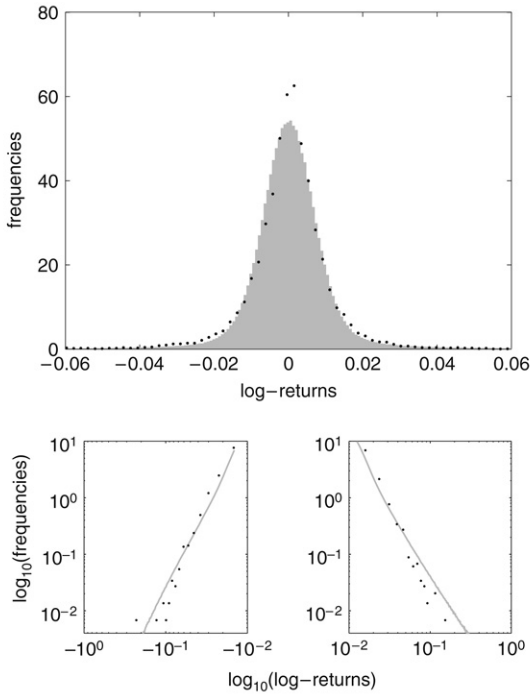

Fig. 5. Top panel: Histogram of daily log-returns of Standard & Poor's 500 index [42] (black) fitted with the L-scheme log-returns limiting distribution f_T^L, Eq. (4.19), (grey). Bottom panel: The corresponding left and right boundary data ranges in the log–log scale. For the fit parameters, see Table 1.

the use of the proposed schemes (compare Fig. 1 with Figs. 4 and 5). Moreover, they suggest that the R-scheme is better in describing the empirical distributions for small values of the log-returns, while the L-scheme is preferred in modeling boundary data ranges (see Figs. 4 and 5). In order to catch in one model different properties of the considered schemes we propose the third, say R/L-scheme, that joins both ideas. Namely, for each scheme let us take its own sequence of the random packet sizes; (M_j^R), (M_j^L), respectively. Then, define the random indices K_n^R, K_n^L, Eqs. (3.11), (3.13), by means of the corresponding packet sizes, and a new, R/L-scheme index $K_n^{R/L}$ as a mixture of K_n^R and K_n^L, i.e.

$$K_n^{R/L} = \begin{cases} K_n^R & \text{with probability } p_0, \\ K_n^L & \text{with probability } 1 - p_0 \end{cases}$$

for some mixing probability $0 \le p_0 \le 1$. In the framework of the introduced mixed R/L-scheme, in analogy to (3.10), (3.12), (3.14) and (3.15), the log-return at time T reads as

$$\log\left(\frac{S^{R/L}(T)}{S_0}\right) = \sum_{k=1}^{K_n^{R/L}} \tilde{X}_{n,k} = \sum_{k=1}^{N_n^{R/L}} X_{n,k},$$

where

$$N_n^{R/L} = \sum_{j=1}^{K_n^{R/L}} M_j = \mu_{K_n^{R/L}}.$$

If the respective packet sizes M_j^R, M_j^L have heavy-tailed distributions, satisfying (4.16) with some $0 < \gamma^R, \gamma^L < 1$ (with γ^R possibly different than γ^L), then the R/L-scheme limiting log-returns distribution is given by the following mixture of the probability density functions f_T^R, Eq. (4.17), and f_T^L, Eq. (4.19),

$$f_T^{R/L}(x, \alpha, \gamma^R, \gamma^L, p_0) = p_0 f_T^R(x; \alpha, \gamma^R) + (1 - p_0) f_T^L(x; \alpha, \gamma^L). \tag{5.21}$$

(Exemplary plots of the probability density function $f_T^{R/L}$ are shown in Fig. 6.) Similarly, the limiting option price results from the formulas (4.18) and (4.20), and it reads as

$$C_0^{R/L}(T; \gamma^R, \gamma^L, p_0) = p_0 C_0^R(T; \gamma^R) + (1 - p_0) C_0^L(T; \gamma^L). \tag{5.22}$$

Continuing the previous example of Standard & Poor's 500 index data [42] we have fitted the limiting distribution $f_T^{R/L}$ proposed in the R/L-scheme with the empirical distribution of the considered data. The results are shown in Fig. 7.

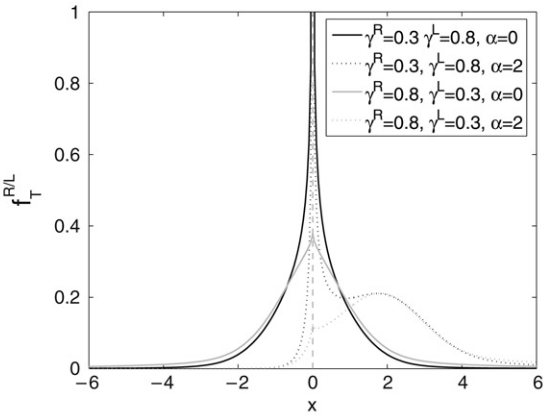

Fig. 6. Exemplary plots of the log-returns limiting probability density function $f_T^{R/L}(x; \alpha, \gamma^R, \gamma^L, p_0)$, Eq. (5.21), in the R/L-scheme with mixing probability $p_0 = \frac{1}{2}$, $(\tilde{\sigma} = 1, T = 1)$.

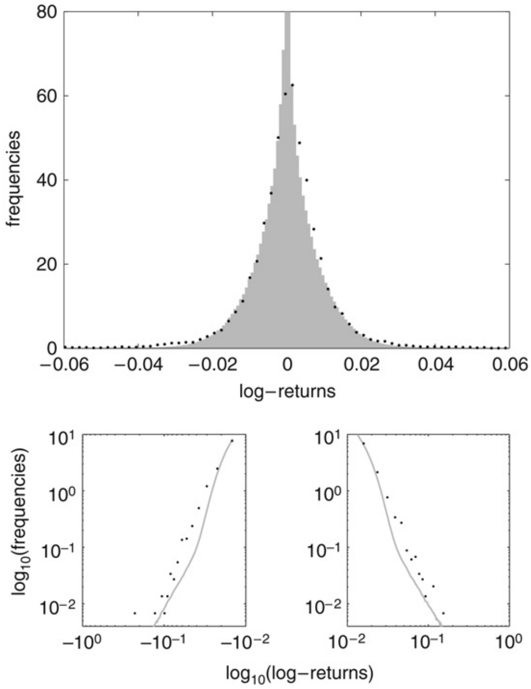

Fig. 7. Top panel: Histogram of daily log-returns of Standard & Poor's 500 index [42] (black) fitted with the R/L-scheme log-returns limiting distribution $f_T^{R/L}$, Eq. (5.21), (grey). Bottom panel: The corresponding left and right boundary data ranges in the log–log scale. For the fit parameters, see Table 1.

Parameters of the fitted distribution (i.e. volatility $\tilde{\sigma} > 0$, the skewness parameter α, the heavy-tail power-law exponents $0 < \gamma^R, \gamma^L < 1$ of the R- and L-scheme waiting/residence times between the cumulative price jumps, and the mixing probability $0 < p_0 < 1$) are given in Table 1. Note that the R/L-scheme log-returns limiting distribution fits the empirical distribution well on whole its support (not only around zero or for the boundary ranges).

The R/L-scheme can be effective also in analysing data from other stock exchanges (i.e. the log-returns of commodity prices or the log-returns of stock indices), as well as in modeling prices of bonds, as can be seen from the following examples. In the first, we concern daily log-returns of NASDAQ value weighted index from years 1990–2005 [42] (see Fig. 8). In the second example we model distribution of monthly log-returns of prices of 7-year bonds from the USA during the calendar period from years 1946–2006 [42] (Fig. 9). Valid returns require a price of the previous and the current periods, and are calculated by dividing the current price plus the interest payments by the previous price. The obtained values of fit parameters are gathered in Table 2.

The presented examples show that the limiting distribution of log-returns proposed in the R/L-scheme can be applied with satisfactory results in modeling wide range of financial data having distributions with high peaks at zero, skewed or with tails heavier than the Gaussian distribution, used in the Black–Scholes model, a limiting case of the CRR scheme.

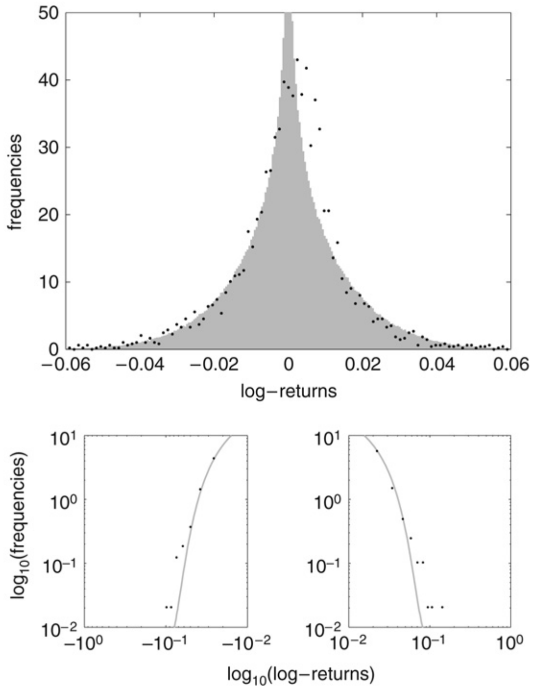

Fig. 8. Top panel: Histogram of daily log-returns of NASDAQ value weighted index [42] (black) fitted with the R/L-scheme log-returns limiting distribution $f_T^{R/L}$, Eq. (5.21), (grey). Bottom panel: The corresponding left and right boundary data ranges in the log–log scale. For the fit parameters, see Table 2.

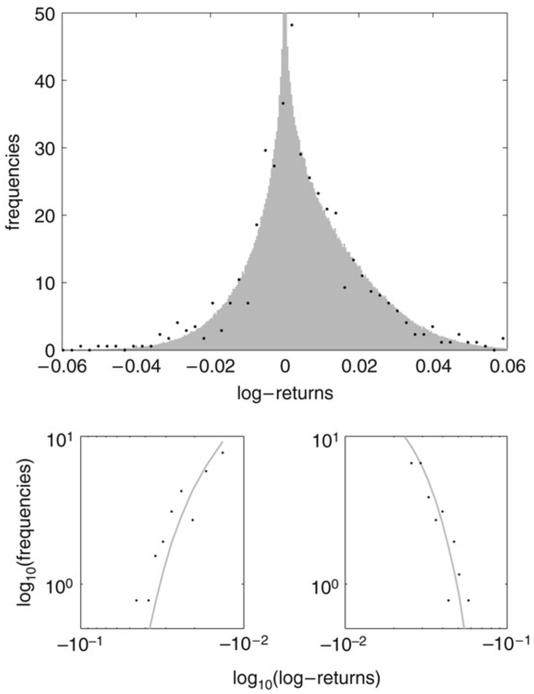

Fig. 9. Top panel: Histogram of log-returns of 7-year bonds [42] (black) fitted with the R/L-scheme log-returns limiting distribution $f_T^{R/L}$, Eq. (5.21), (grey). Bottom panel: The corresponding left and right boundary data ranges in the log–log scale. For the fit parameters, see Table 2.

6. Conclusions

In this paper, we have proposed and studied a generalization of the RR asset-pricing model based on the CTRW notion. The obtained theoretical results have been fitted with several sets of financial data and appeared to be useful for analysis of

Table 2
Parameters of the log-returns limiting distributions (obtained in the R/L scheme) fitted with the empirical distribution of the log-returns of NASDAQ index and of 7-year bonds.

Data	γ^R	γ^L	α	$\tilde{\sigma}$	p_0	T
NASDAQ	0.45	0.74	0.292	0.4012	0.98	1 day
Bonds	0.55	0.75	0.1116	0.0675	0.98	1 month

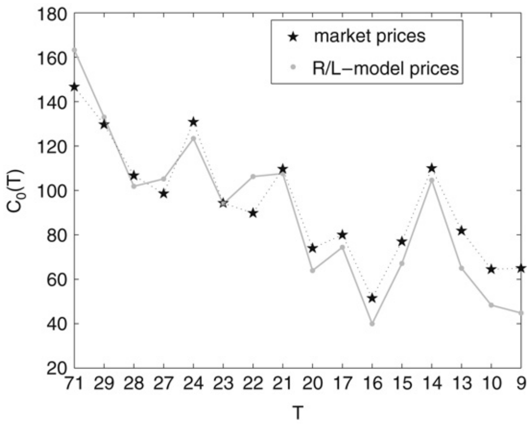

Fig. 10. European call option prices for 100 units of Standard & Poor's 500 index from the time period 23.07.2008–19.09.2008 [42] with strike price $K = 1150$ and $r = 4\%$ (black stars) compared with the corresponding calibrated prices $C_0^{R/L}(T; \gamma^R, \gamma^L, p_0)$, formula (5.22), for parameters $\gamma^R, \gamma^L, p_0, \alpha, \tilde{\sigma}$ as in Table 1 (grey solid line).

the data. The model therefore provides some explanation of the observed departure from the Gaussian statistics (appearing in the classical Black–Scholes formula), alternative to that based directly on the theory of Lévy-stable laws. It yields the limiting log-returns distribution (5.21) that is a mixture of the Gaussian and a generalized arcsine distributions and better describes the empirical distribution of the financial data. The corresponding pricing formula (5.22) for the European call option is an essential extension of the famous Black–Scholes formula, and it may provide a satisfactory description of the option market prices. Indeed, a comparison of the theoretical option prices and the real market data, presented in Fig. 10, suggests the possible application of the results obtained in the framework of the proposed R/L scheme for forecasting the option prices.

The introduced coupled CTRW-like form of the random number of price changes in the RR model results from the concept of the random clustering procedure, proposed and studied in Refs. [5,16], and assumes some agglutination of the successive price ups and downs in the clusters of random sizes. The applied clustering schemes are shown to be connected with different hedging strategies. Such an approach provides quite realistic description of the asset price on the real market where indeed we observe randomizations of the value of successive groups of jumps. Moreover, it indicates applicability of the CTRW idea, used mainly in physics to model anomalous diffusion, to describe financial markets, as well.

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