

# ORNSTEIN–UHLENBECK PROCESS WITH NON-GAUSSIAN STRUCTURE\*

JAKUB OBUCHOWSKI<sup>†</sup>, AGNIESZKA WYŁOMAŃSKA<sup>‡</sup>

Diagnostics and Vibro-Acoustics Science Laboratory  
Institute of Mathematics and Computer Science, Hugo Steinhaus Center  
Wrocław University of Technology, Wrocław, Poland

(Received April 11, 2013)

In this paper, we examine the Ornstein–Uhlenbeck process, *i.e.* one of the most famous example of continuous time models. Because many studies indicate that the classic version of the Ornstein–Uhlenbeck process is insufficient to description of examined phenomenon, there is a need to consider various modifications of the conventional process. We introduce generalized versions of the classic process in which the standard Brownian motion (Wiener process) is replaced by  $\alpha$ -stable and variance gamma processes. We analyze similarities and differences between Gaussian and considered non-Gaussian versions of the Ornstein–Uhlenbeck process. We point at testing and estimation procedures which we illustrate by simulated data. In order to illustrate theoretical results, we examine a real financial data set in the context of presented methodology.

DOI:10.5506/APhysPolB.44.1123

PACS numbers: 05.10.–a, 02.50.–r

## 1. Introduction

The Ornstein–Uhlenbeck (OU) process is one of the most famous example of continuous time models. The classic version of the OU process was introduced in [1] as a proper system that can be used to model data with Gaussian and diffusion behavior. From a physical point of view, the process is a stationary solution for the classic Klein–Kramers dynamics [2, 3]. In economics, the Ornstein–Uhlenbeck process is known as the Vasiček model because of the fundamental paper [4], where author proposed to use it as

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\* Presented at the XXV Marian Smoluchowski Symposium on Statistical Physics, “Fluctuation Relations in Nonequilibrium Regime”, Kraków, Poland, September 10–13, 2012.

<sup>†</sup> jakub.obuchowski@pwr.wroc.pl

<sup>‡</sup> agnieszka.wylomanska@pwr.wroc.pl

a model of interest rate time series. It was also proposed to description of currency exchange rates and commodity prices [5, 6]. For other applications, see for example [7]. Moreover, the Ornstein–Uhlenbeck process is very important in theoretical mathematical and physical studies [8, 9].

However, the assumption of normality in the classic version of OU process seems not to be reasonable in number of examined phenomena. Therefore, the Gaussian distribution in the Ornstein–Uhlenbeck model is replaced by its various modifications. One of the simplest modification is to incorporate the  $\alpha$ -stable distribution [10, 11]. Processes based on stable distribution are very useful in modeling data that exhibit fat tails. For example, the classic Ornstein–Uhlenbeck process was extended to the stable case and analyzed in [8] as a suitable model for financial data description. Instead of using Gaussian or  $\alpha$ -stable distribution in the OU process, it is also possible to introduce other distributions more appropriate for observed time series. In this paper, we propose to analyze the variance gamma distribution known also as a generalized Laplace. It has semi-heavy tails, heavier than Gaussian, but well-defined moment generating function categorizes variance gamma distribution as a light-tailed. This distribution is characterized by leptokurticity and possible asymmetry. Those features cause the variance gamma distribution to be useful, for example in modeling stock prices [12–14]. Other modifications of the classic OU process one can find, for instance, in [15]. In this paper, we examine the Ornstein–Uhlenbeck process driven by  $\alpha$ -stable and variance gamma distribution. We indicate similarities between the classic OU model and its modified versions, and point at testing and estimation procedures in considered cases. Motivation of using  $\alpha$ -stable or variance gamma distribution comes from applications in finance, therefore, we illustrate the theoretical results by analysis of real financial data.

The rest of the paper is organized as follows: in Sec. 2, we introduce the classic Ornstein–Uhlenbeck process and its generalized version based on Lévy processes. In Sec. 3, we consider  $\alpha$ -stable and variance gamma processes. Moreover, we examine modified versions of the classic Ornstein–Uhlenbeck process based on those distributions. Next section contains description of testing and estimation procedures for considered models. In Sec. 5, we examine a real financial data set in the context of presented methodology. Last section contains conclusions.

## 2. The classic Ornstein–Uhlenbeck process

In this section, we introduce the classic Ornstein–Uhlenbeck model and its modification based on Lévy processes. The origin of the classic OU process was in 1930, when G.E. Uhlenbeck and L.S. Ornstein showed relation between velocity of a Brownian particle and the normal distribution [1]. Four

decades later, in the 1970s, Vasiček proposed a stochastic financial model, where the interest rate was modeled by the OU process [4]. The classic Ornstein–Uhlenbeck process  $\{X_t\}_{t \geq 0}$  is defined as a solution of stochastic differential equation of the following form

$$dX_t = \theta(\mu - X_t) dt + \sigma dB_t, \quad (1)$$

where  $\mu \in \mathbb{R}$ ,  $\theta > 0$ ,  $\sigma > 0$ , and  $\{B_t\}_{t \geq 0}$  is the Brownian motion (called also the Wiener process). Parameter  $\mu$  represents a long-term mean of the OU process,  $\theta$  is a value of mean-reverting speed, and  $\sigma$  corresponds to the deviation of stochastic factor [1, 4].

But for real phenomena processes based on the Gaussian distribution are often insufficient [17–19]. The idea of generalization of the OU process is to modify a random factor (*i.e.* product of  $\sigma$  and the Wiener noise  $dB_t$ ) in (1). It would affect possibilities of modeling processes with the non-Gaussian structure. It is known that the Itô's integral is well-defined for semimartingales, particularly for Lévy processes [20, 21]. Therefore, a stochastic differential equation that defines generalized OU process can be written as follows

$$dX_t = \theta(\mu - X_t) dt + dL_t, \quad (2)$$

where  $\{L_t\}_{t \geq 0}$  denotes a general Lévy process (*i.e.* process with independent stationary increments) while parameters  $\theta$  and  $\mu$  have the same meaning as in equation (1).

### 3. Non-Gaussian Lévy processes

As it was mentioned in Sec. 2, Gaussian models are insufficient in certain applications. Therefore, in this section, we introduce two non-Gaussian Lévy processes which can be used in the generalized version of OU process presented in (2).

#### 3.1. $\alpha$ -stable Lévy process

A random variable  $X$  is said to be  $\alpha$ -stable ( $X \sim S_\alpha(\sigma, \beta, \mu)$ ) if its characteristic function has the following form [23]

$$\phi_X(z) = \begin{cases} \exp \left\{ -\sigma^\alpha |z|^\alpha \left( 1 - \nu\beta (\operatorname{sgn}(z)) \operatorname{tg} \frac{\pi\alpha}{2} \right) + \nu\mu z \right\} & \alpha \neq 1 \\ \exp \left\{ -\sigma |z| \left( 1 + \nu\beta \frac{2}{\pi} (\operatorname{sgn}(z)) \ln |z| \right) + \nu\mu z \right\} & \alpha = 1 \end{cases},$$

where  $0 < \alpha \leq 2$ ,  $\sigma \geq 0$ ,  $-1 \leq \beta \leq 1$  and  $\mu \in \mathbb{R}$ . Two most known instances of  $\alpha$ -stable distribution are Gaussian ( $\alpha = 2$ ) and Cauchy ( $\alpha = 1$ ,  $\beta = 0$ ). The main reason of using  $\alpha$ -stable noise in stochastic models is a heavy tail feature, *i.e.* extreme values are much more probable than for normally distributed random variables.

The  $\alpha$ -stable Lévy process  $\{L_{\alpha,\sigma,\beta}(t)\}_{t \geq 0}$  is a process with independent and identically distributed increments of  $S_\alpha\left(\sigma t^{\frac{1}{\alpha}}, \beta, 0\right)$  distribution on the interval of length  $t$ . Therefore, the stochastic differential equation which defines an  $\alpha$ -stable OU process has the following form

$$dX_{\alpha,\sigma,\beta}(t) = \theta(\mu - X_{\alpha,\sigma,\beta}(t)) dt + dL_{\alpha,\sigma,\beta}(t).$$

Three sample trajectories of the process  $\{L_{\alpha,\sigma,\beta}(t)\}_{t \geq 0}$  are presented in Fig. 1. One can see jumps typical for  $\alpha$ -stable models. A lower value of  $\alpha$  evokes higher jumps. For more properties of  $\alpha$ -stable Lévy processes, see for instance [10, 11, 23].

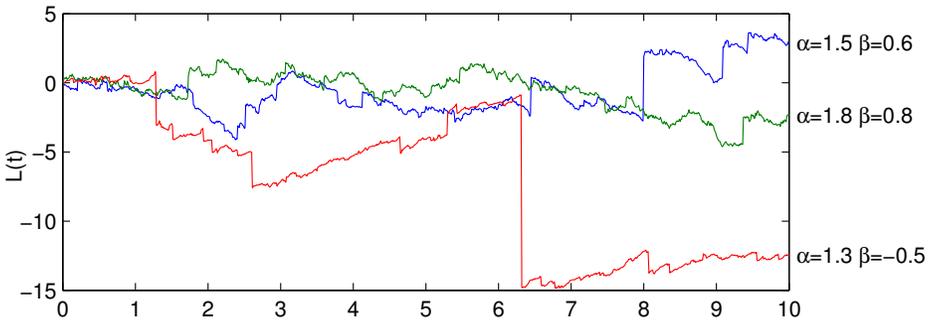


Fig. 1. Sample trajectories of the  $\alpha$ -stable Lévy processes with  $\mu = 0$ ,  $\sigma = 1$  and different  $\alpha$  and  $\beta$  parameters.

### 3.2. Variance gamma Lévy process

The next Lévy process we introduce is the generalized Laplace motion (GLM). It is also known as the variance gamma process (VG), named after its representation as a gamma-subordinated Brownian motion [24]. GLM is a Lévy process with independent and identically distributed increments of generalized Laplace distribution (equivalently — variance gamma distribution). We say that GLM is standard if it is a zero-mean process with variance equal to  $t$  on interval of length  $t$ . A random variable  $X$  has the generalized Laplace distribution ( $X \sim \mathcal{GL}(\delta, \mu, \sigma, \nu)$ ) if its characteristic function is given by [25]

$$\phi_X(z) = e^{i\delta z} \left(1 - i\mu z + \frac{\sigma^2 z^2}{2}\right)^{-\frac{1}{\nu}}, \quad \sigma, \nu > 0, \quad \mu, \delta \in R. \quad (3)$$

In the literature, there are several equivalent parameterizations of the VG distribution. They might allow to control some properties of the distribution (*e.g.* moments) or simplify certain estimation methods [25]. The parameterization used in formula (3) leads to a compact form of the generalized Laplace probability density function

$$f_{\delta,\mu,\sigma,\nu}(x) = \frac{\sqrt{2} \left( \frac{|x-\delta|}{\sqrt{\mu^2+2\sigma^2}} \right)^{\frac{1}{\nu}-\frac{1}{2}}}{\sqrt{\pi}\sigma\Gamma\left(\frac{1}{\nu}\right)} \exp\left(\frac{(x-\delta)\mu}{\sigma^2}\right) K_{\frac{1}{\nu}-\frac{1}{2}}(y),$$

where  $y = \frac{|x-\delta|\sqrt{\mu^2+2\sigma^2}}{\sigma^2}$  and  $K_p(x)$  is modified Bessel function of the order of  $p$  [26]. The existence of moment generating function causes finiteness of all moments. Formulas for mean, variance, skewness and kurtosis are as follows [25]

$$EX = \frac{\mu}{\nu} + \delta, \quad E(X - EX)^3 = \mu\sqrt{\nu} \frac{2\mu^2 + 3\sigma^2}{(\mu^2 + \sigma^2)^{\frac{3}{2}}}, \quad (4)$$

$$\text{Var } X = \frac{\mu^2 + \sigma^2}{\nu}, \quad E(X - EX)^4 = 6\nu - \frac{3\nu\sigma^4}{(\mu^2 + \sigma^2)^2} + 3. \quad (5)$$

In Fig. 2, we show trajectories of GLM with various parameters. One can observe bigger jumps and longer periods of small fluctuations if kurtosis is higher ( $\nu = 1$ ). Distribution of the generalized Laplace motion in  $t \geq 0$  is  $\mathcal{GL}(\delta t, \mu, \sigma, \frac{\nu}{t})$  [25]. For zero-mean GLM, we write  $\text{GLM}_{\mu,\sigma,\nu}$ . Now, we can write stochastic differential equation which defines the Ornstein–Uhlenbeck process with generalized Laplace structure

$$dX_{\mu,\sigma,\nu}(t) = \theta(\mu - X_{\mu,\sigma,\nu}(t)) dt + \sigma d\text{GLM}_{\mu,\sigma,\nu}(t).$$

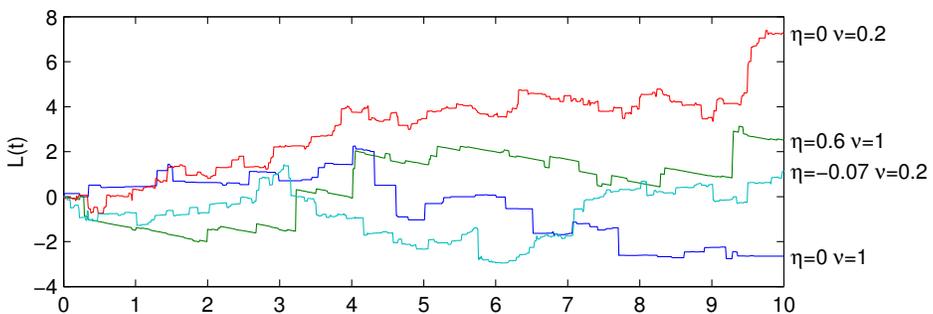


Fig. 2. Sample trajectories of the standard generalized Laplace motion with different  $\mu$  and  $\nu$  parameters.

Figure 3 contains trajectories of both considered here OU processes with the non-Gaussian structure. We can see a higher fluctuation of the sample mean in the  $\alpha$ -stable case than for other processes. The sample mean of the generalized Laplace OU process behaves very similar to the mean of the classic OU process even though GLM has jumps and Brownian motion has not.

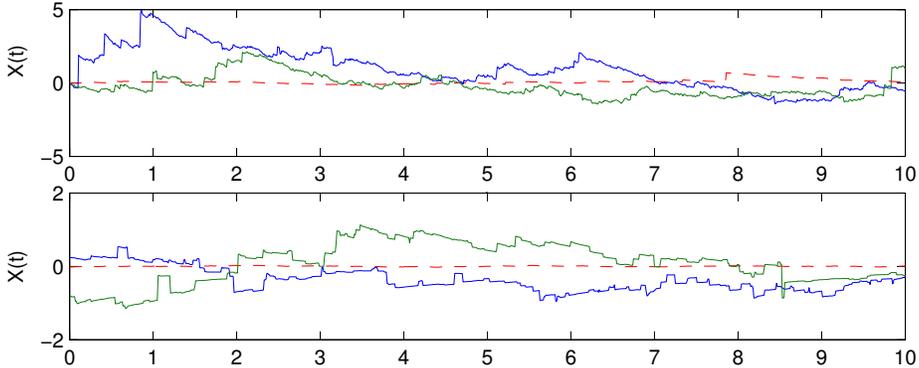


Fig. 3. Sample mean and trajectories of generalized OU processes with  $\theta = 0.6$ ,  $\mu = 0$ :  $\alpha$ -stable (top panel) with  $\alpha = 1.5$ ,  $\sigma = 1$ ,  $\beta = 0.6$  and standard generalized Laplace with  $\mu = 0.06$ ,  $\nu = 0.36$  (bottom panel). Dashed (red) lines denote sample means obtained from 1000 trajectories of those processes.

#### 4. Testing and estimation

In this section, we describe estimation procedure for generalized OU processes. Moreover, we test this procedure by using Monte Carlo simulations.

If observations are equally spaced in domain (*e.g.* time domain), we treat them as a time series. We mention the discrete version of OU process is simply autoregressive (AR) time series of the order of 1. In order to check if the data represents autoregressive time series, we should examine its stationarity and order of autocorrelation. For testing stationarity, we propose the quantile lines test [27, 28]. This procedure is described in the next paragraph. This method can be used only when lots of trajectories are available, for example, when the experiment is repeated many times and, as a result, we obtain many realization of the same process. In the case of one trajectory, we can use, for example, the test for regime variance described in details in [29]. The order of autoregressive model can be examined by using autocorrelation (ACF) and partial autocorrelation functions (PACF). In the case of an  $\alpha$ -stable distribution, the ACF and PACF are not defined but even in this case those functions can be useful in the problem of AR

model order detection [30]. The next step is to estimate parameters  $\mu$  and  $\theta$  in (2) using the sample mean for  $\mu$  and the Whittle estimator for  $\theta$  [31]. Another method of estimating  $\theta$  is the Yule–Walker method [16]. It involves finiteness of variance of analyzed time series, hence we cannot apply it to the  $\alpha$ -stable OU process.

Once estimators of  $\mu$  and  $\theta$  from OU model are calculated, we check if residuals can be treated as a vector of independent random variables. Finally, we can fit parameters of suspected distributions. In this paper, we examine three distributions: Gaussian,  $\alpha$ -stable and generalized Laplace. Final step is to check how effectively information criteria choose the best model.

In order to check this procedure, we have simulated 5000 trajectories of each of OU processes with the same parameters  $\mu$  and  $\theta$  and length equal to 4000. Values of parameters are:  $\mu = 0$ ,  $\theta = 0.85$  (OU process parameters),  $\sigma = 3.6 \times 10^{-3}$  (classic OU process parameter),  $\alpha = 1.6$ ,  $\beta = -0.1$  ( $\alpha$ -stable OU process parameters) and  $\mu = -0.0001$ ,  $\sigma = 0.0033$ ,  $\nu = 0.84$  (OU process with zero-mean GL noise parameters).

As it was mentioned, there is a visual test for stationarity called “quantile lines”.  $N$  trajectories of the process  $\{X_t\}_{t \in T}$  correspond to  $N$  realizations of a random variable  $X_t$  for each  $t$ . To verify if distributions of each  $X_t$  are the same, one can calculate their sample quantiles and check if they change in time. Plotting quantiles of each  $X_t$  leads to so-called quantile lines. See Fig. 4 for quantile lines plotted for 3 different OU processes with 5000 trajectories each. One can see parallelism in every plot which means stationarity in every case. Large deviation of extreme quantiles estimators is caused by quantity of sample trajectories. The next step of checking the estimation

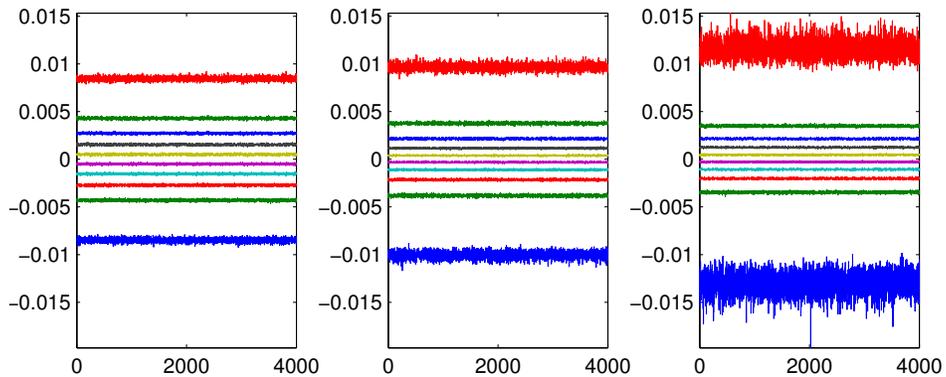


Fig. 4. Quantile lines obtained from 5000 trajectories of classical (left panel), generalized Laplace (center panel) and  $\alpha$ -stable OU processes.

procedure is to examine ACF and PACF. Expected behavior of PACF for AR(1)-type time series is a spike on the lag 1 and zeros on higher-order lags. ACF should decrease exponentially [32]. ACF and PACF for one trajectory of each process are presented in Fig. 5. If the data represents AR(1)-type

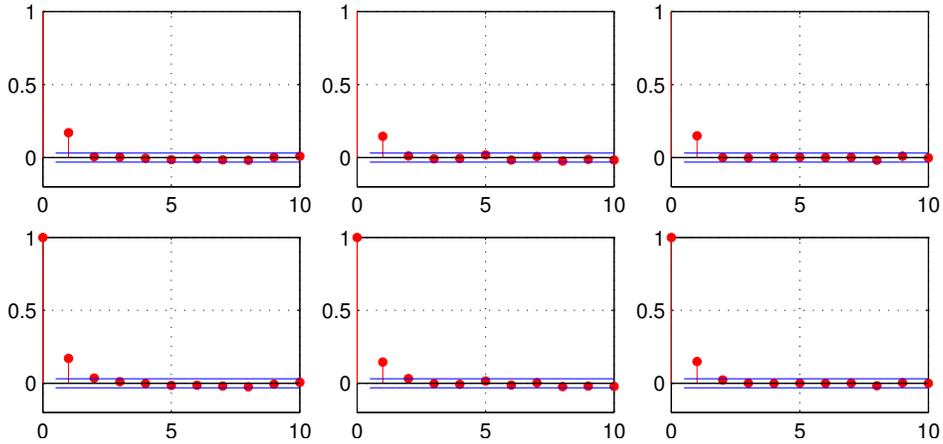


Fig. 5. ACF (bottom) and PACF (top) for Gaussian (left panels), GL (center panels) and  $\alpha$ -stable OU (right panels) processes.

time series, one can estimate parameters of the OU process given in (2). We can firstly estimate  $\mu$  as a sample mean of one trajectory and subtract it from the data. Thus we have constructed zero-mean time-series for which the Whittle estimator of  $\theta$  can be applied. Figure 6 shows deviation of estimators of  $\mu$  and  $\theta$  calculated for all of three analyzed OU processes. Semi-heavy tail of the GL distribution does not affect more deviation than

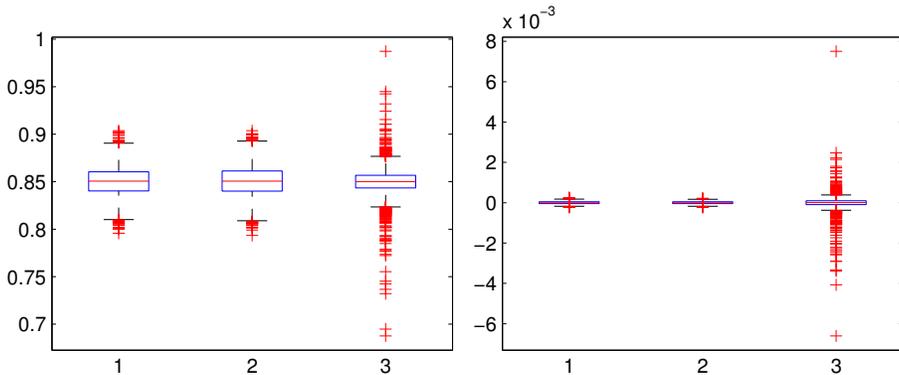


Fig. 6. The boxplots of  $\mu$  (left panel) and  $\theta$  (right panel) estimated from 5000 sample trajectories of Gaussian (1), GL (2) and  $\alpha$ -stable OU (3) processes.

in the light-tail Gaussian case. The  $\alpha$ -stable OU process is characterized by much higher deviation of  $\hat{\mu}$ , but behavior of  $\hat{\theta}$  is specific. Extreme values are more scattered, but interquartile range is smaller than in other cases. For every trajectory of three processes, we have constructed residual series. One series of each type is presented in Fig. 7. Even in one sample trajectory of length 4000, the  $\alpha$ -stable OU process is characterized by sporadic large values of residuals. The generalized Laplace residuals have more frequent, but smaller deviations. Recall that variance of  $\alpha$ -stable is infinite and variances for noise of GL and Gaussian structures are the same but in the case of VG distribution, we observe leptokurticity.

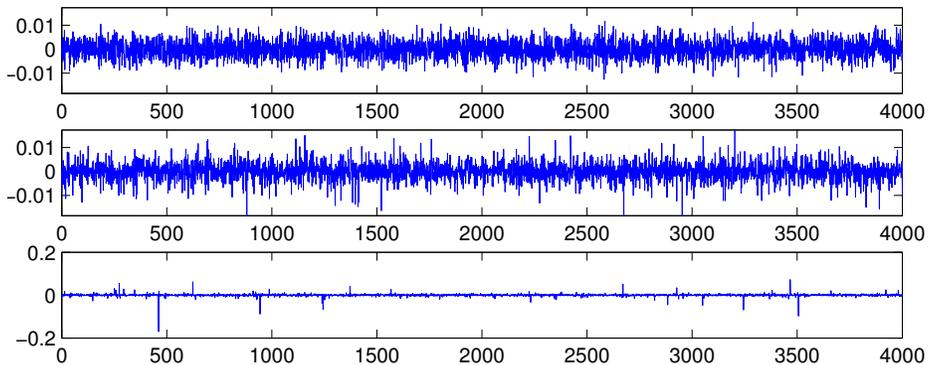


Fig. 7. Residuals obtained from simulated trajectories of Gaussian (top panel), generalized Laplace (middle panel) and  $\alpha$ -stable (bottom panel) OU processes.

When 5000 vectors of residuals from every of three types OU processes are obtained, we can fit their distributions. For the Gaussian and GL distribution, we use maximum likelihood estimators (MLE) [26]. Starting points for parameters  $\delta$ ,  $\mu$ ,  $\sigma$  and  $\nu$  in GL distribution are calculated using the method of moments (MM) [25]. Estimators of  $\alpha$ ,  $\sigma$ ,  $\beta$  and  $\mu$  in  $\alpha$ -stable distribution are calculated using the regression method [23] implemented in MFE Toolbox [33].

Once we have all of parameters estimated, we compute values of the Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) [34, 35] for every of 15 000 simulated trajectories and check if minimal values are realized by the true model. As Table I shows, information criteria are faultless when residuals are GL or  $\alpha$ -stable distributed and we check fit of three analyzed distributions. If the proper model is Gaussian, minimum values of AIC and BIC are obtained by fitted Gaussian model in 4733, GL model in 260 and  $\alpha$ -stable model in 7 of 5000 cases. This indicates that the estimation procedure used here is almost unerring.

TABLE I

Efficiency of information criteria.

		Proper model		
		Gaussian	GL	$\alpha$ -stable
Fitted model	Gaussian	94.66%	0%	0%
	GL	5.20%	100%	0%
	$\alpha$ -stable	0.14%	0%	100%

## 5. Applications

In order to check usage of models examined in Sec. 4, we fit them to a real-life financial data. The data represents S&P500 index stock quotes after every trading hour from the period 05.06.2009–09.20.2011 and contains 4160 quotes (Fig. 8). Since it looks non-stationary, we decided to compute hourly log-returns (Fig. 9). After this transformation, neither trend nor periodicity can be observed.

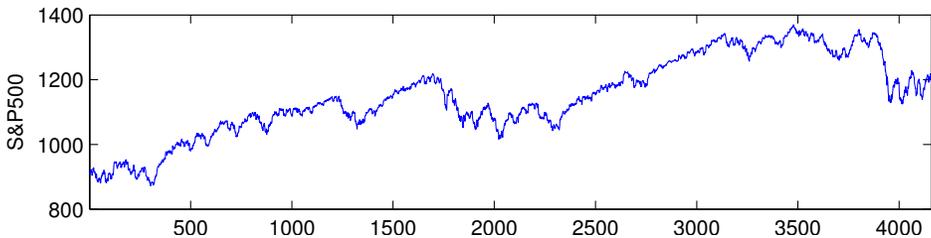


Fig. 8. S&amp;P500 data from 05.06.2009 to 09.20.2011 (4160 observations).

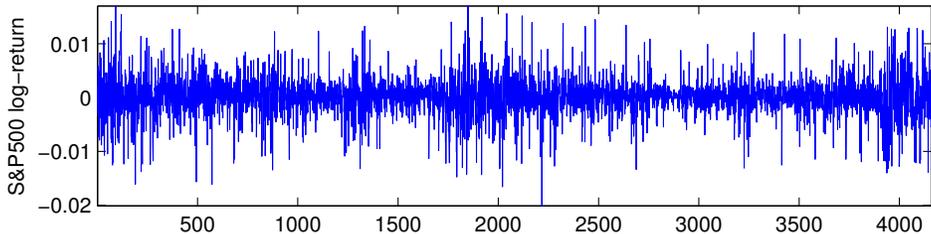


Fig. 9. Hourly log-returns of dataset.

The first step is to examine ACF and PACF. Plots of these functions show possibility of fitting OU process to the data (Fig. 10). One can observe spike of PACF for lag 1 with almost zeros in subsequent lags. ACF for lag 1 is about 0.15, so further values should be close to 0.

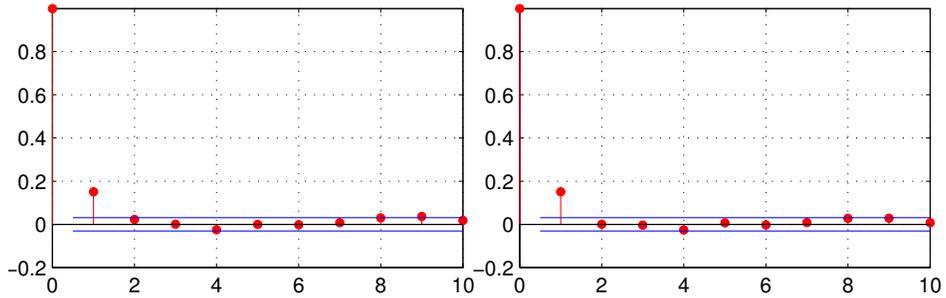


Fig. 10. ACF (left panel) and PACF (right panel) of S&P500's hourly log-returns.

Estimated values of  $\mu$  and  $\theta$  are  $7.32 \times 10^{-5}$  and 0.85, respectively. Vector of residuals and corresponding ACF and PACF are shown in Fig. 11. See no autocorrelation there. Moreover, vector of residuals appears very similar to vector of GL residuals and completely different than  $\alpha$ -stable (Fig. 7).

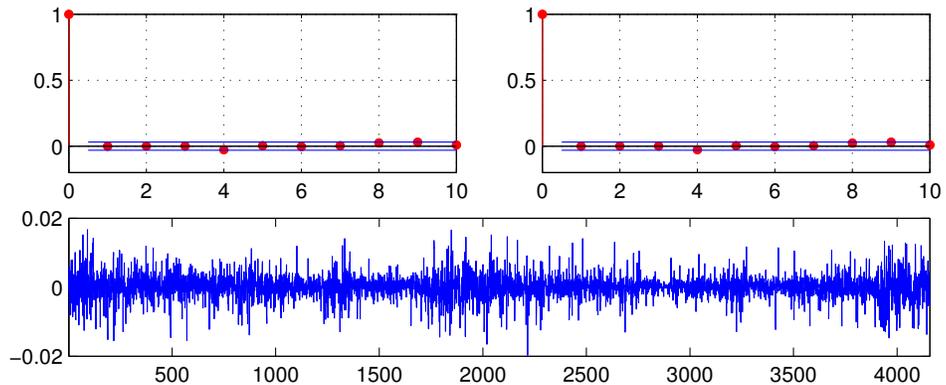


Fig. 11. Vector of residuals (bottom panel) with ACF (top left panel) and PACF (top right panel) of it.

Table II shows estimated parameters for each of three analyzed types of noise with Kolmogorov–Smirnov statistics and corresponding  $p$ -values [36]. At the significance level 0.05, we cannot reject hypothesis of two distributions of residuals: generalized Laplace and  $\alpha$ -stable. Other tests have to be used for distribution identification due to large differences between  $\alpha$ -stable residuals (Fig. 7) and residuals obtained from S&P500 data (Fig. 11). Quantile–quantile (QQ) plots show better fit of the GL distribution (Fig. 12). The

TABLE II

Estimated parameters with Kolmogorov–Smirnov test results.

	Parameter value		K–S statistic value	$p$ -value
$\mathcal{N}(\mu, \sigma)$	$\mu$	0	0.0694	$7.3629 \times 10^{-18}$
	$\sigma$	0.003619		
$\mathcal{GL}(\delta, \mu, \sigma, \nu)$	$\delta$	0.000149	0.0196	0.0801
	$\mu$	-0.000125		
	$\sigma$	0.003256		
	$\nu$	0.841245		
$S_\alpha(\sigma, \beta, \mu)$	$\alpha$	1.623035	0.0139	0.3972
	$\sigma$	0.001882		
	$\beta$	-0.131476		
	$\mu$	-0.000003		

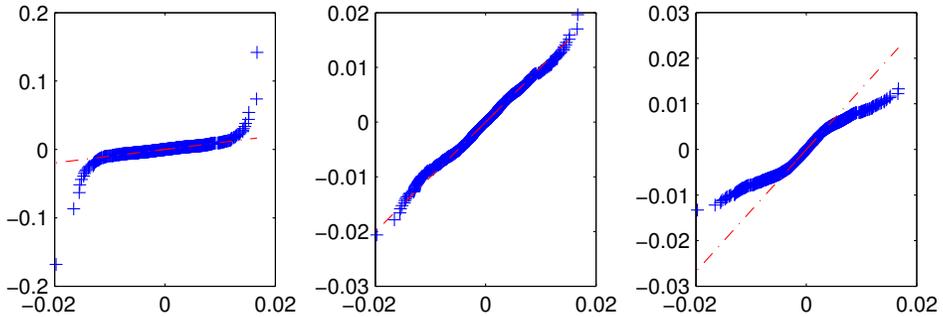


Fig. 12. QQ plots of  $\alpha$ -stable (left panel), generalized Laplace (center panel) and Gaussian (right panel) distributions fitted to residuals.

$\alpha$ -stable distribution has much heavier tails. For comparison, QQ plot for fitted (and rejected by Kolmogorov–Smirnov test at the level 0.05) Gaussian distribution shows its unacceptably light tail. The last comparison criteria we use are AIC and BIC. Obtained values are shown in Table III. Indisputably, the best model for hourly log-returns of S&P500 is the Ornstein–Uhlenbeck process with generalized Laplace distribution.

TABLE III

Information criteria — Akaike information criterion (AIC) and Bayesian information criterion (BIC). Lower value means better fit.

	AIC	BIC
$\mathcal{N}(\mu, \sigma)$	-34939.7464	-34908.2762
$\mathcal{GL}(\delta, \mu, \sigma, \nu)$	-35552.5235	-35521.0532
$S_\alpha(\sigma, \beta, \mu)$	-35480.1272	-35448.6569

## 6. Conclusions

In this paper, we have examined one of the most famous example of the continuous time models, *i.e.* the Ornstein–Uhlenbeck process. Because the classic version of the process seems not to be reasonable in the number of examined phenomenon, then we have introduced its modification. We have combined the well-known process with the  $\alpha$ -stable and variance gamma distributions. We have pointed at the similarities and differences between examined systems. Moreover, we have presented in details testing and estimation procedures. Motivation for using  $\alpha$ -stable and variance gamma distributions comes from financial applications thus the theoretical results we have illustrated by the analysis of real financial time series. We believe that proposed models can be useful tools in the advanced real data analysis.

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